PRELIMINARY EXAMINATION IN ANALYSIS

Part I, Real Analysis

August 20, 2012

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1. Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that a measurable function $f : X \to [0, \infty)$ is integrable if and only if $\sum_{n=0}^{\infty} \mu\big(\{x \in X : f(x) \geq n\}\big)$ converges.

2. If $f(x, y) \in \mathrm{L}^2(\mathbb{R}^2)$, show that $f(x + x^3, y + y^3) \in \mathrm{L}^1(\mathbb{R}^2)$.

3. Let μ be a measure in the plane for which all open squares are measurable, with the property that there exists $\alpha \geq 1$, such that if two open squares Q and Q' are translates of each other and their closures $\bar{Q}$ and $\bar{Q}'$ have a non-empty intersection, then

$$\mu(\bar{Q}) \leq \alpha\, \mu(Q') < \infty\,.$$

(For Lebesgue $\alpha = 1$, in general $\alpha \geq 1$.) Show that horizontal lines have zero measure.

4. Let $\{A_{\vec{k}}\}$ be any sequence of real numbers, indexed by vectors $\vec{k} = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$. Suppose that f is a positive integrable function on $\mathbb{R}^n$: $f > 0$ and $\int f\, dx < \infty$.

a) Show that $\liminf\limits_{|\vec{k}| \to \infty} \displaystyle\int \big|\cos\big(\vec{k} \cdot x + A_{\vec{k}}\big)\big| f(x)\, dx \;>\; 0\,,$ where $|\vec{k}| = \sum\limits_i |k_i|$.

Let $\{r_{\vec{k}}\}$ be any sequence of positive real numbers, such that $\sum\limits_{\vec{k}} r_{\vec{k}} \big|\cos\big(\vec{k} \cdot x + A_{\vec{k}}\big)\big| < \infty$.

b) Show that $\displaystyle\sum_{\vec{k}} r_{\vec{k}} < \infty\,.$