PRELIMINARY EXAMINATION IN ANALYSIS

Part II, Complex Analysis

August 20, 2012

---

1. Let f and g be entire analytic functions, and assume that $|f(z)| < |g(z)|$ whenever $|z| > 1$. Show that f/g is a rational function.

2. Assume that f is analytic outside the disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and takes its values inside this disk. Prove that $|f'(2)| \leq \frac{1}{3}$.

3. Suppose that $\{f_n\}$ is a sequence of analytic functions on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ that is Cauchy with respect to the $\mathrm{L}^2(D)$ metric. Show that $\{f_n\}$ converges uniformly on compact subsets of D to an analytic function $f : D \to \mathbb{C}$.

4. Prove that for any simply connected open subset of the complex plane, there exists an analytic function that cannot be analytically continued to a larger open domain.