PRELIMINARY EXAMINATION: APPLIED MATHEMATICS I

August 22, 2012, 1:00-2:30

Work all 3 of the following 3 problems.

1. For $f \in L^2(0,1)$, define

$$(Tf)(x) = \int_0^x f(s)\,ds \qquad \text{for } x \in (0,1).$$

(**a**) Show that T is a bounded linear operator on $L^2(0,1)$, and that $Tf \in C^0(0,1)$.

(**b**) Consider a sequence of functions f_n bounded in $L^2(0,1)$. Show that there exist a subsequence n_p and a function $f \in L^2(0,1)$ such that $(Tf_{n_p})(x)$ converges pointwise to $(Tf)(x)$ for every x in $(0,1)$. [Hint: Use the weak convergence.]

(**c**) Show that T is compact in $L^2(0,1)$.

2. Uniform Boundedness.

(**a**) State the Uniform Boundedness Principle.

(**b**) Assume that x belongs to a normed space X and that for some $c \in [0,\infty)$ we have that $|l(x)| \le c\|l\|$ for all $l \in X^*$. Show that $\|x\| \le c$.

(**c**) Suppose that $S \subset X$ is such that

$$\sup_{x \in S} |l(x)| < \infty \qquad \text{for all } l \in X^*.$$

Such sets S are called *weakly bounded*. Show that S is bounded in X.

3. Let H be a Hilbert space, let $S \in B(H,H)$, and let $T \in B(H,H)$ be self-adjoint.

(**a**) Prove that if S is bounded below, then S is one-to-one and its range is closed in H.

(**b**) Prove that the point spectrum of T is real.

(**c**) Prove that $\lambda \in \rho(T)$, the resolvent, if and only if $T_\lambda = T - \lambda I$ is bounded below.