PRELIMINARY EXAMINATION: APPLIED MATHEMATICS II

August 22, 2012, 2:40-4:10

Work all 3 of the following 3 problems.

1. Consider the Telegrapher's equation

$$u_{tt} + 2u_t + u = c^2 u_{xx} \qquad \text{for } x \in \mathbb{R} \text{ and } t > 0;$$

with $u(x;0) = f(x)$ and $u_t(x;0) = g(x)$ given in $L^2(\mathbb{R})$.

(**a**) Use the Fourier transform (in x only) and its inverse to find an explicit representation of the solution.

(**b**) Justify that your representation is indeed a solution.

(**c**) Show that the solution behaves like the solution of a damped wave equation; that is, the solution is a wave packet that is moving to the right with a given constant speed, and a wave packet that is moving to the left with the same speed. What determines these wave packets and their speed and damping rate?

2. Given $I = [0, b]$, consider the problem of finding $u : I \to \mathbb{R}$ such that

$$\begin{cases} u'(s) = g(s) f(u(s)), & \text{for a.e. } s \in I, \\ u(0) = \alpha, \end{cases} \tag{1}$$

where $\alpha \in \mathbb{R}$ is a given constant, $g \in L^p(I)$, $p \geq 1$, and $f : \mathbb{R} \to \mathbb{R}$ are given functions. We suppose f is Lipschitz continuous and satisfies $f(0) = 0$.

(**a**) Consider the functional

$$F(u) = \alpha + \int_0^s g(\sigma)\, f(u(\sigma))\, d\sigma.$$

Show that F maps $C^0(I)$ into $C^0(I) \cap W^{1,p}(I)$. Moreover, show that $u \in C^0(I) \cap W^{1,p}(I)$ satisfies (1) if and only if it is a fixed point of F.

(**b**) Show that (1) has a unique solution $u \in C^0(I) \cap W^{1,p}(I)$ for any $g \in L^p(I)$ and $b > 0$.

3. Let Ω be a domain with a smooth boundary. Consider the differential problem

$$\begin{aligned} p - \nabla \cdot a\nabla p - \nabla \cdot b\nabla q + d(p - q) &= 0 \quad \text{in } \Omega, \\ -\nabla \cdot c\nabla q + d(q - p) &= f \quad \text{in } \Omega, \\ -(a\nabla p + b\nabla q) \cdot \nu &= g \quad \text{on } \partial\Omega, \\ q &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where a, b, c, and $d \geq 0$ are bounded, smooth functions, $f \in H^{-1}(\Omega)$, and $g \in H^{-1/2}(\partial\Omega)$. Moreover, assume that there is some $\gamma > 0$ such that $a \geq \gamma$, $c \geq \gamma$, and $|b| \leq \gamma$.

(**a**) Define a suitable variational problem for the differential equations. Be sure to identify your function spaces for p, q, and the test functions.

(**b**) Show that there is a unique solution to the variational problem.