# Numerical Analysis Exam: Part B, August 2012

1. The following ordinary differential equation and corresponding predictor-corrector (P-C) method are given,

$$
\begin{aligned}
&y' = f(y), \quad t > 0; \quad y(0) = y_0 \\
&y^*_{n+1} = y_n + hf(y_n), \quad n = 1,2,.. \\
&y_{n+1} = y_n + h\left(f(y_n) + f(y^*_{n+1})\right)/2 \\
&\left(y_n \approx y(t_n), \quad t_n = nh, \quad n = 1,2,..\right)
\end{aligned}
$$

(a) Determine the order of the approximation and whether the method is stable.
(b) Determine the region of absolute stability.
(c) Describe how step size control can be based on this P-C technique.

(2) Consider the elliptic partial differential equation,

$$
\begin{aligned}
&-(a(x,y)u_x)_x - (a(x,y)u_y)_y + b(x,y)u = f(u), \quad (x,y) \in \Omega \\
&a(x,y) \ge a > 0,\ b(x,y) \ge b > 0, \quad \Omega = \left\{(x,y), 0 < x < 1, 0 < y < 1\right\} \\
&Boundary\ \partial\Omega:\ u = 0\ for\ y = 0,1, \quad 0 \le x \le 1, \\
&\qquad\qquad\qquad u_x = \alpha u\ \ for\ x = 0,1, \quad 0 < y < 1
\end{aligned}
$$

(a) Reformulate this boundary value problem on weak form and describe a finite element approximation based on triangulation of $\Omega$ and piecewise polynomials.
(b) Prove that the bilinear and linear forms related to the weak formulation satisfies the standard conditions for convergence when $\alpha = 0, f(u) \equiv 0$.
(c) Present solution methods for the nonlinear algebraic system resulting from the discretization in part (a) above.

3. Approximate the partial differential equation below,

$$
\begin{aligned}
&u_t + Au_x = f(x,t), \quad 0 < x < 1, t > 0, \\
&u(x,0) = u_0(x), \quad 0 \le x < 1, \quad periodic\ boundary\ conditiond
\end{aligned}
$$

(a) Define the standard upwind approximation of the equation when A is a real number and determine its order of approximation.
(b) Prove that this scheme is stable and satisfies a maximum principle under suitable condition on $\Delta t/\Delta x$ (the step sizes) when $A > 0, \alpha = 0, f(x,y) \equiv 0$.
(c) Use von Neumann analysis to prove stability in $L^2$ when *A* is a symmetric matrix and the upwind scheme is replaced by Lax Friedrich. Express the stability condition (CFL condition) in terms of the spectral radius of *A*.