THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

The Preliminary Examination in Probability
Part I

Aug 23, 2012

Problem 1 (30pts)**.** Let X and Y be two independent random variables with $X + Y \in \mathbb{L}^1$. Show that $X \in \mathbb{L}^1$.

Problem 2 (35pts)**.** For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, let $\mathcal{G}^{\perp\!\!\perp}$ be the **independent complement** of $\mathcal{G}$, i.e.,

$$\mathcal{G}^{\perp\!\!\perp} = \{X \in \mathcal{L}^0(\mathcal{F}) \,:\, X \text{ is independent of } \mathcal{G}\}.$$

Show that independence behaves differently than orthogonality (absence of correlation) by constructing an example of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ such that

(1) there exists $A \in \mathcal{F} \setminus \mathcal{G}$ with $0 < \mathbb{P}[A] < 1$, and
(2) $\mathcal{G}^{\perp\!\!\perp}$ contains only a.s.-constant random variables.

Problem 3 (35pts)**.** Let the stochastic process $\{X_n\}_{n \in \mathbb{N}_0}$ be constructed inductively as follows:

- $X_0 = 0$, and
- for $n \in \mathbb{N}$, and conditionally on $\mathcal{F}_{n-1} = \sigma(X_0, \dots, X_{n-1})$, we set
  - if $X_{n-1} = 0$, then $X_n = -1, 0$ or 1, with probabilities $\frac{1}{2n}$, $1 - \frac{1}{n}$ and $\frac{1}{2n}$, respectively,
  - if $X_{n-1} \neq 0$, then $X_n = nX_{n-1}$ or 0 with probabilities $\frac{1}{n}$, $1 - \frac{1}{n}$.

Show that $\{X_n\}_{n \in \mathbb{N}}$ is a martingale which converges to 0 in probability but not a.s.