THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

Preliminary Examination in Probability
Part II

August, 2012

Problem 2.1. (30 points) Let B a standard, one-dimensional Brownian motion. Denote by

$$Z_t = e^{B_t - \frac{1}{2}t},\ T_b = \inf\{t > 0 |\ Z_t = b\} \text{ and } Z^* = \sup_{t \geq 0} Z_t.$$

Compute

(1) $\mathbb{P}[T_b < \infty]$ for $b > 1$
(2) the law of Z^* and the law of $1/Z^*$

Problem 2.2. (30 points) Let X be a continuous semi-martingale, and X^n a sequence of continuous processes of bounded variation such that, for each t, we have

$$\mathbb{P}[\lim_{n\to\infty} X^n_t = X_t] = 1.$$

If $f : \mathbb{R} \to \mathbb{R}$ is a function of class C^1, show that

$$\lim_{n\to\infty} \int_0^t f(X^n_s) dX^n_s = \int_0^t f(X_s) dX_s + \frac{1}{2} \int_0^t f'(X_s) d\langle X \rangle_s,$$

holds $\mathbb{P}$-a.s. for every $t \geq 0$.

Problem 2.3. (40 points, Brownian bridge) Let $(B_t)_{0\leq t\leq 1}$ a standard one-dimensional Brownian motion (with time horizon $T = 1$) and denote by $(\mathcal{F}_t)_{0\leq t\leq T}$ the (augmented) filtration generated by B. Denote by

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1),\quad 0 \leq t \leq 1,$$

a new filtration (where the value of the BM at terminal time $T = 1$ is known at time t). Show that

(1)
$$\mathbb{E}[B_t - B_s | \mathcal{G}_s] = \frac{t-s}{1-s}(B_1 - B_s)$$

(2) the process $(\beta_t)_{0\leq t\leq 1}$, defined by

$$\beta_t = B_t - \int_0^t \frac{B_1 - B_s}{1-s} ds,$$

is a $\mathcal{G}_t$-Brownian motion, independent of B_1

(3) Denote by

$$X^x_t = xt + B_t - tB_1,\quad 0 \leq t \leq 1,$$

(the Brownian bridge ending at x at time $T = 1$).
Show that

$$X^x_t = \int_0^t \frac{x - X^x_s}{1-s} ds + \beta_t,\quad 0 \leq t \leq 1.$$

Note: one can obtain the representation of the Brownian bridge in (3) from (2) by conditioning the BM on its teminal value.