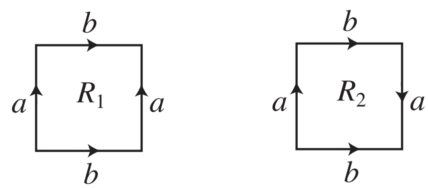
Preliminary Examination in Topology: August 2012

Algebraic Topology portion

Instructions: Do all three questions.

Time Limit: 90 minutes.

1. Let R_1 be a rectangle and identify opposite edges to give a torus T. Let R_2 be a rectangle and identify opposite edges to give a Klein bottle K. Let S be the space gotten by further identifying the a-edge of R_1 with the a-edge of R_2, and the b-edge of R_1 with the b-edge of R_2 as indicated in the figure below.



(1) Use a Mayer-Vietoris sequence to compute the first and second homology groups of S. You may take the homology groups of 1 and 2-manifolds as facts.
(2) Is there a deformation retraction of S onto K?
(3) Is T a retract of S?

2. Thinking of S^1 as the unit circle in R^2, let $f : S^1 \to S^1$ be such that $f(-x) = -f(x)$.
(1) Let C be the quotient of S^1 under the equivalence relation $x \sim -x$ for $x \in S^1$ and let $p : S^1 \to C$ be the quotient map. Then f induces a map $g : C \to C$ so that $p \circ f = g \circ p$. Show that g induces an injection on fundamental groups.
(2) Show that f is not homotopic to a constant map.
(3) Consider S^3 as the unit sphere in R^4. Use (2) to show that there is no map $h : S^3 \to S^1$ such that $h(-x) = -h(x)$.

3.
(1) What are the spaces that 2-fold cover the Mobius band?
(2) Let X be the connected sum of three projective planes. Show that there is an orientable surface Y that 2-fold covers X by drawing a picture of the covering space and covering map. Show that, up to equivalence of covers, there is a unique 2-fold covering of X by an orientable surface. (Hint: Show that the covering corresponds to the kernel of a unique map on fundamental group, by picking the right generators of $\pi_1(X)$).