Preliminary Examination in Topology: August 2012

Differential Topology portion

Instructions: Do all three questions.

Time Limit: 90 minutes.

1a. Let $f\colon X \to Y$ be a submersion, and let $Z \subset Y$ be a submanifold (with $\partial X, \partial Y, \partial Z$ empty). Prove that $f^{-1}(Z)$ is a manifold.

b. Recall that one definition of $\mathbb{R}P^k$ is as the quotient of S^k by the antipodal map, with smooth structure defined so that the projection $p\colon S^k \to \mathbb{R}P^k$ is a local diffeomorphism. Suppose that k is odd and $Z_1, Z_2 \subset \mathbb{R}P^k$ are compact submanifolds of positive dimension for which the oriented intersection number $I(Z_1, Z_2)$ is defined. Prove $I(Z_1, Z_2) = 0$. (*Hint:* What conditions are guaranteed by well-definedness of $I(Z_1, Z_2)$?) Is the corresponding statement true for the mod-2 intersection number I_2?

2. On $\mathbb{R}^2$, let $\omega = (\sin^4 \pi x + \sin^2 \pi(x+y))dx - \cos^2 \pi(x+y)\, dy$.

a. Show that ω is closed.

b. Let η be the unique 1-form on the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ such that $p^*\eta = \omega$, where $p\colon \mathbb{R}^2 \to T^2$ is projection. The parametrized curve $\gamma\colon \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(\theta) = (2\theta, -3\theta)$ is a line whose image $C \subset T^2$ is an oriented circle. Prove that $\int_C \eta \neq 0$. What is its sign?

c. Is η closed? Exact?

d. There is a standard embedding $i\colon T^2 \to S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ (determined by the formula $i \circ p(x,y) = (\cos 2\pi x, \sin 2\pi x, \cos 2\pi y, \sin 2\pi y)$). Is there a closed form ζ on $\mathbb{R}^4$ for which $i^*\zeta = \eta$? Justify your answer.

3. For the unit sphere $S^n \subset \mathbb{R}^{n+1}$, let $f\colon S^n \to S^n$ be the map reversing the signs of all but one coordinate,

$$f(x_0, x_1, \ldots, x_n) = (x_0, -x_1, \ldots, -x_n).$$

a. Compute the Lefschetz number $L(f)$.

b. For which values of n is f homotopic to a map without fixed points?