Preliminary Examination in Algebra–Fall semester

January 11, 2013, RLM 9.166, 1:00-2:30 p.m.

Do three of the following five problems.

1. Let G be a group.

(i) Show that if every nontrivial element of G has order 2 then G is abelian.

(ii) Show that (a) fails if we replace 2 by any larger prime p.

2. Prove that all groups of order less than 60 are solvable.

3. Let n be an integer greater than 3. Classify up to isomorphism all groups which arise as semidirect products of $\mathbb{Z}/2^n\mathbb{Z}$ by $\mathbb{Z}/2\mathbb{Z}$.

4. Assume that S is an integral domain, and $R \subseteq S$ is a subring containing the identity element. Recall that an element a in S is *integral* over R if there exists a monic polynomial $f(x)$ in $R[x]$ such that f is not identically zero and $f(a) = 0$. Then the integral closure of R in S is the subset of elements in S that are integral over R. An integral domain is called *integrally closed* if it is equal to its integral closure within its field of fractions.

(i) Show that a is integral over R if and only if $R[a]$ is a finitely generated R-submodule of S.

(ii) Suppose that R is a unique factorization domain. Prove that R is integrally closed.

(iii) Is the ring

$$\{a + b\sqrt{-3} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}\}$$

integrally closed? Give a proof to justify your answer.

5. Let A be an n by n matrix with entries in $\mathbb{C}$. Recall that for $\lambda \in \mathbb{C}$, the generalized λ-eigenspace V_λ of A is the set of all vectors $v \in \mathbb{C}^n$ such that, for some m, one has $(A - \lambda)^m v = 0$.

(i) Show that $V_\lambda \neq 0$ if, and only if, λ is a root of the characteristic polynomial of A.

(ii) Show that $\mathbb{C}^n$ is the direct sum of the spaces V_λ, as λ runs over the roots of the characteristic polynomial of A.