Preliminary Examination in Algebra–Spring semester

January 11, 2013, RLM 9.166, 2:40-4:10 p.m.

Do three of the following five problems.

1. Suppose F is a field and D is an integral domain containing F, which is finite-dimensional as a vector space over F. Show that D is a field.

2. Let p be a prime number, k a field, and $\beta \neq 0$ an element of k.

(i) Prove that either the polynomial $f(x) = x^p - \beta$ is irreducible in $k[x]$, or there exists an element α in k such that $\beta = \alpha^p$.

(ii) Assume that $f(x)$ is irreducible in $k[x]$ and k contains a primitive pth root of unity. Show that $k(\alpha)$ is a splitting field for $f(x)$.

3. Show that the following polynomials are irreducible over the given rings.

(i) The polynomial $x^4y^3 + 3x^5y + x^6 + x^4y + 27x^2 + xy + 3y + 6$ in $\mathbb{Q}[x, y]$.

(ii) The polynomial $x^6y + x^2y^2 - y^2 + x^2 + 3x + 2$ in $\mathbb{C}[x, y]$.

4. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - x^2 - 1$.

(i) Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_8 of order 8.

(ii) Give a complete description of the lattice of subfields of K.

5. Let L/K be a separable extension of fields.

(i) Assume that for some integer $N \geq 1$ the inequality $[K(\alpha) : K] \leq N$ holds for all elements α in L. Prove that $[L : K] \leq N$.

(ii) Show that if L/K is a separable, infinite, normal extension of fields then $\mathrm{Aut}(L/K)$ is an infinite group.