PRELIMINARY EXAMINATION IN ANALYSIS

Part I, Real Analysis

January 9, 2013

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1. Let $f \in \mathrm{L}^\infty(\mu)$ be a nonnegative bounded μ-measurable function. Consider the set R_f consisting of all positive real numbers w such that $\mu\big(\{x : |f(x) - w| \leq \varepsilon\}\big) > 0$ for every $\varepsilon > 0$.

(a) Prove that R_f is compact.

(b) Prove that $\|f\|_{\mathrm{L}^\infty} = \sup R_f$.

2. Let $f, f_1, f_2, \ldots$ be functions in $\mathrm{L}^1\big([0,1]\big)$ such that $f_k \to f$ pointwise almost everywhere. Show that $\|f_k - f\|_1 \to 0$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$, such that $\left|\int_A f_k\right| < \varepsilon$ for all k and all measurable set $A \subset [0,1]$ with measure $|A| < \delta$.

3. Let $p > 0$, and denote by $\mathrm{L}^p_{\text{weak}}(\mathbb{R})$ the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$N_p(f) = \sup_{\alpha > 0} \alpha^p \big|\{x \in \mathbb{R}^n : |f(x)| > \alpha\}\big|$$

is finite. Prove that the simple functions are <u>not</u> dense in $\mathrm{L}^p_{\text{weak}}(\mathbb{R})$, in the sense that there exists a function $f \in \mathrm{L}^p_{\text{weak}}(\mathbb{R})$ such that $N_p(f - h_k) \to 0$ fails to hold for every sequence of simple functions $h_1, h_2, \ldots$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with compact support, and let $g \in \mathrm{L}^1(\mathbb{R})$. Prove that $f * g$ is absolutely continuous on $\mathbb{R}$.