PRELIMINARY EXAMINATION: APPLIED MATHEMATICS I

January 7, 2013, 1:00–2:30 p.m.

Work all 3 of the following 3 problems.

1. Let H be a real Hilbert space and suppose that P is a bounded linear projection on H. Let $Q = I - P$ and define $M = P(H)$ and $N = Q(H)$.

(**a**) Show that there exists $C > 0$ such that

$$\|x - Px\| \leq C \inf_{y \in M} \|x - y\| \quad \text{for all } x \in H.$$

(**b**) Prove that P is an orthogonal projection if and only if

$$\inf_{\substack{y \in N, \|y\|=1 \\ x \in M}} \|y - x\| = 1.$$

[Hint: For the converse, it is enough to show that for any $z \in H$, $z - Pz = Qz \perp M$. Consider $y = Qz/\|Qz\|$.]

2. Let X and Y be separable and reflexive Banach spaces and let $T \in B(X, Y)$. Suppose that $y_n \in Y$ and $y_n \overset{w}{\rightharpoonup} y$ (y_n converges weakly to y), and that there are $x_n \in X$ such that

$$Tx_n = y_n \quad \text{for all } n.$$

Moreover, suppose that $\|x_n\| \leq M$ for some $M > 0$.

(**a**) State the Generalized Heine-Borel Theorem for a separable and reflexive Banach space.

(**b**) Show that for some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$, $Tx_{n_k} \overset{w}{\rightharpoonup} y$.

(**c**) If T is a compact operator, show that the convergence in (**b**) is strong for the entire subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

3. Let H be a complex Hilbert space and $T \in C(H, H)$ a compact linear operator.

(**a**) Show that T^*T is compact, self-adjoint, and positive (i.e., nonnegative) on H.

(**b**) Show that if $w \in N(T^*T)$, then in fact $Tw = 0$.

(**c**) Show that there are a pair of orthonormal sets $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ such that for any $x \in H$,

$$Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, u_n \rangle v_n,$$

where $\{\lambda_n^2\}_{n=1}^{\infty}$ are the *positive* eigenvalues of T^*T and $\{u_n\}_{n=1}^{\infty}$ are the corresponding eigenvectors.