Numerical Analysis Exam: Part A, January 2013

1. Consider a real $m \times n$ matrix A.

(a) Describe how singular value decomposition can be used to solve the least squares problem $\min_x \|Ax - b\|$.
(b) Describe two methods for computing the vector u that minimizes,

$$\min_{u \neq 0} \frac{\langle u, Au \rangle}{\langle u, u \rangle}.$$

2. Let $f(x): R^n \to R^n$ be a smooth function such that $f(a) = 0$ and the matrix $(\partial f / \partial x)$ is full rank.

(a) Write down the Newton method for solving $f(x) = 0$.
(b) Show that the Newton method converges to the root a if the initial condition x_0 is sufficiently close to a.
(c) Suppose that the following iteration is used instead

$$x_{n+1} = x_n - A f(x_n),$$

where A is a $n \times n$ matrix. Find a sufficient condition on A such that this iteration is also convergent for x_0 close to a.

3. In the numerical integration formula

$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(c),$$

(a) if constants a, b, c can be chosen arbitrarily, what is the highest degree k such that this formula is exact for all polynomials of degree up to k?
(b) Find the constants a, b, c for which the formula is exact for all polynomials of degree up k.