The University of Texas at Austin
Department of Mathematics

The Preliminary Examination in Probability
Part I

Jan 10, 2013

Problem 1 (35pts)**.** Let $\{X_n\}_{n\in\mathbb{N}}$ be a uniformly integrable sequence of nonnegative random variables, and let $M_n = \max(X_1, \dots, X_n)$, for $n \in \mathbb{N}$. Show that

$$\lim_n \tfrac{1}{n}\mathbb{E}[M_n] = 0.$$

(*Hint:* For $c \geq 0$, express $M_n \mathbf{1}_{\{M_n > c\}}$ using $X_k \mathbf{1}_{\{X_k > c\}}$, $k = 1, \dots, n$.)

Problem 2 (30pts)**.** Consider the following two statements about the random variable X:

(a) X is discrete, i.e., there exists a countable set $C \subseteq \mathbb{R}$ such that $\mathbb{P}[X \in C] = 1$.

(b) The characteristic function φ_X of X is periodic, i.e., there exists $T > 0$ such that $\varphi_X(t+T) = \varphi_X(t)$, for all $t \in \mathbb{R}$.

Show that $(b) \Rightarrow (a)$. Is it true that $(a) \Rightarrow (b)$?

Problem 3 (35pts)**.** Let $X_n = \sum_{k=1}^n \xi_k$, $n \in \mathbb{N}$, $X_0 = 0$, be a simple random walk, i.e., $\{\xi_n\}_{n\in\mathbb{N}}$ are iid with $\mathbb{P}[\xi_1 = 1] = 1 - \mathbb{P}[\xi_1 = -1] = p \in (0,1)$. Furthermore, let $g : \mathbb{Z} \to \mathbb{R}$ be a function such that $\{g(X_n)\}_{n\in\mathbb{N}_0}$ is a submartingale.

For which $p \in (0,1)$ does it necessarily follow that g is convex, i.e., that $g(k+1) - 2g(k) + g(k-1) \geq 0$, for all $k \in \mathbb{Z}$.