Preliminary Examination in Topology: January 2013

Differential Topology portion

Instructions: Do all three questions. Note that there are two pages.

Time Limit: 90 minutes.

1a. Given $n, d \in \mathbb{Z}^+$ and $a_0, \ldots, a_n \in \mathbb{R} - \{0\}$, prove that the solution set in $\mathbb{R}^n$ of the equation $a_0 = a_1 x_1^d + \cdots + a_n x_n^d$ is a manifold. What is its dimension?

b. Let $p\colon \mathbb{R}^3 - \{0\} \to \mathbb{R}P^2$ be the usual projection. Prove that the solution set in $\mathbb{R}^3 - \{0\}$ of $x^d + y^d = z^d$ has the form $p^{-1}(X_d)$ for some subset X_d of $\mathbb{R}P^2$.

c. Prove that X_d is a manifold. (*Hint:* Local coordinates.)

d. Compute (with full justification) the mod 2 intersection number $I_2(X_d, L)$, where $L \subset \mathbb{R}P^2$ is the projective line given by setting $x = 0$.

e. For each $d = 1, 2, 3$, determine whether the inclusion map $i\colon X_d \to \mathbb{R}P^2$ is homotopic to a constant map.

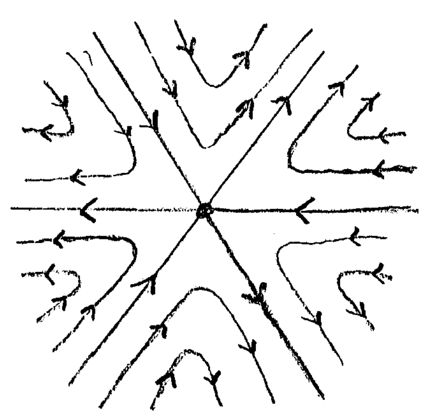
2. On $\mathbb{R}^3$, let $\omega = y dx \wedge dz$.

a. Compute $d\omega$ and $\int_{S^2} \omega$, where S^2 is the unit sphere centered at the origin, oriented as the boundary of the ball.

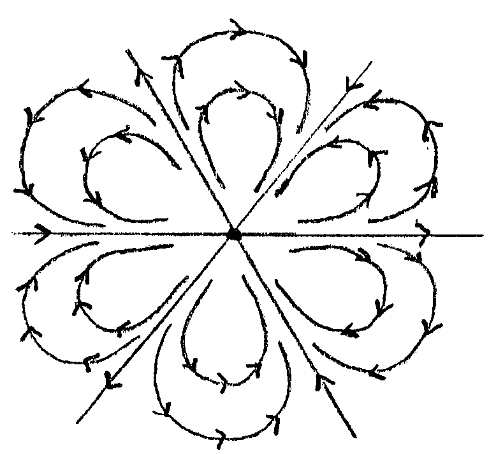
b. Let $f\colon F \to S^2$ be a smooth map of degree n from a closed, oriented surface F. Compute $\int_F f^*\omega$.

c. For which values of n does there exist a compact, oriented 3-manifold X with oriented boundary given by a surface F having a map as in (b), and a closed 2-form $\eta \in \Omega^2(X)$ whose restriction to F is $f^*\omega$? Prove that your answer is complete.

3a. Pictured below are two isolated zeros p, q of a vector field in $\mathbb{R}^2$. Compute the index of each.



p



q

b. Suppose a vector field $\mathbf{v}$ on $\mathbb{R}^n$ has exactly two isolated zeros p, q, and p, q are connected by a flow-line of the vector field. Furthermore, assume one can modify the vector field in a compact neighborhood of the flow line and merge p, q to give new vector field $\mathbf{w}$ with a single isolated zero r. Show that the index of r must be the sum of the indices of p and q. (An example in $\mathbb{R}^2$ is pictured below.)