ALGEBRA PRELIMINARY EXAM: PART I

Problem 1

Fix an $n \times n$ matrix A with entries in an algebraically closed field K. Let V be the space of $n \times n$ matrices over K that commute with A. Observe that V is a vector space over K. Show that

$$\dim V \geq n,$$

and the equality holds if and only if the characteristic polynomial of A equals the minimal polynomial of A.

Problem 2

A finite group G is *supersolvable* if there is an increasing chain of subgroups

$$\{1_G\} = G_0 \subset G_1 \subset \ldots \subset G_r = G$$

such that each G_i is normal in G, and G_{i+1}/G_i is cyclic for all i.

(a) Show that every p-group is supersolvable.

(b) Give an example of a solvable group that is not supersolvable.

Problem 3

Let A be the ring of $n \times n$ matrices over a field F.

(a) Show that for any subspace V of F^n, the set I_V of matrices whose kernel contains V is a left ideal of A.

(b) Show that every left ideal of A is principal.

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