PRELIMINARY EXAM IN ANALYSIS
PART I – REAL ANALYSIS.

AUGUST 26, 2013 – TIME: 9:00–10:30.

Name (**print**): ____ UT ID: ____________

Please write clearly, and staple your work with the signed exam sheet on top !

Problem 1

(a) Provide an example of a sequence of measurable functions on $[0,1]$ which converges in L^1 to the zero function but does not converge pointwise a.e.

(b) Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of integrable functions on $[0,1]$ such that $\|f_n\|_{L^1} \leq n^{-2}$ holds for all n. Show that $\{f_n\}_{n=1}^{\infty}$ converges pointwise a.e. to the zero function.

Problem 2

Let $(x_1, x_2, \ldots)$ be an arbitrary sequence of real numbers in $[0,1]$ (possibly dense). Show that the series $\sum_k k^{-3/2}|x - x_k|^{-1/2}$ converges for almost every $x \in [0,1]$.

Problem 3

Assume that μ is a finite Borel measure on $\mathbb{R}^n$, and that there exists a constant $0 < R < \infty$ such that the k-th moments of μ satisfy the bound

$$\int d\mu(x)|x|^k < R^{k^r} \quad \forall k \in \mathbb{N}\,,$$

for some $0 < r \leq 1$. Prove that μ has bounded support contained in $\{x \in \mathbb{R}^n : |x| \leq R\}$ if $r = 1$, and in $\{x \in \mathbb{R}^n : |x| \leq 1\}$ if $0 < r < 1$.

Problem 4

Let f be a continuous function on $[0,1]$. Find

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\, dx.$$

Justify your answer.