PRELIMINARY EXAMINATION: APPLIED MATHEMATICS -Part I

August 23, 2013, 1:00-2:30

Work all 3 of the following 3 problems.

1. Let T be a bounded linear operator on a Banach space X such that the sequence $x, Tx, T^2x, \ldots$ is bounded for every $x \in X$. Show that if $\lambda \in \sigma(T)$, then $|\lambda| \leq 1$.
(Hint: Investigate the limit when N goes to infinity of $R_N x = \sum_{n=0}^{N} T^n x/\lambda^n$.)

2. Consider an operator $A : L^2[0,1] \to L^2[0,1]$ defined by

$$Af(x) = \int_0^x f(t)\, dt.$$

(**a**) Show that $A^* f(x) = \int_x^1 f(t)\, dt$ and $A^*Af(x) = \int_0^1 [1 - \max(x,t)]\, f(t)\, dt$.

(**b**) Show that A^*A is self-adjoint, positive, and compact on $L^2[0,1]$.

(**c**) Show that if $\lambda \neq 0$ is an eigenvalue of A^*A with eigenfunction f, then $\lambda f'' = -f$ almost everywhere on $[0,1]$, and also $f'(0) = 0$ and $f(1) = 0$.

(**d**) Show that $||A^*A|| = 4/\pi^2$ and $||A|| = 2/\pi$.

3. Let H be a Hilbert space and $T \in B(H,H)$.

(**a**) If $\|T\| \leq 1$, show that for all $y \in H$, there is a unique solution to $T^2x - Tx - 12x = y$. [Hint: $T^2 - T - 12I = (T+3I)(T-4I)$.]

(**b**) If T is self-adjoint and positive, show that $I + T$ is invertible.

(**c**) If T is self-adjoint and there is some $\gamma > 0$ such that $(Tx, x) \geq \gamma\|x\|^2$ for all $x \in H$, show that T is invertible.