Part I

1. Consider the linear least squares problem

$$\min_x \|Ax - b\|_2, \tag{1}$$

where $A \in R^{m \times n}$ with $m \geq n$.

   (a) Derive the normal equations for solving (1).

   (b) Show how to use QR decomposition and SVD (singular value decomposition) to solve (1).

   (c) Suppose A does not have full column rank. Is the least squares solution unique? Characterize all solutions in terms of the SVD of A.

   (d) Suppose A does have full column rank, but many of its singular values are small (for example, $m = 100, n = 50, \sigma_1 = 2, \sigma_1, \cdots, \sigma_{25} > 1$ and $\sigma_{26}, \ldots, \sigma_{50} < 10^{-13}$). How will you solve the least squares problem (1) in this case? Discuss.

2. Consider $g(x) = \frac{1}{2}x + 2x^2 - \frac{3}{2}x^3$ and the iteration befined by $x_{n+1} = g(x_n)$.

   (a) Show that for any $x_0 \in [0, 1]$, the sequence x_n converges. For each n, x_n is a function of the initial value x_0, and we denote such dependence as $x_n(x_0)$. Find the limit function $g_\infty(x_0) = \lim_{n\to\infty} x_n(x_0)$ for $x_0 \in [0, 1]$.

   (b) For each $x_0 \in [0, 1]$, determine the order of convergence of $\{x_n(x_0)\}$.

3. Let a continuous function $f : [a, b] \to \mathbb{R}$ and a non-negative, integrable weight function $w : [a, b] \to \mathbb{R}$ be given, where $w(x) = 0$ at only finitely many points. For any given $n \geq 0$, let Π_n denote the space of polynomials of degree at most n, and consider the problem of finding $p_n \in \Pi_n$ to minimize the fitting error

$$E[p_n] = \int_a^b w(x)[p_n(x) - f(x)]^2 \, dx.$$

   (a) Show that if E is minimized by $p_n(x) = \sum_{k=0}^n c_k x^k$, then $c = (c_0, \ldots, c_n)$ must necessarily satisfy $Ac = F$ for an appropriate $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and $F \in \mathbb{R}^{n+1}$.

   (b) Show that A is symmetric, positive-definite. Moreover, show the polynomial p_n^* with coefficient vector $c^* = A^{-1}F$ minimizes E over Π_n, that is $E[p_n^*] \leq E[p_n]$ for all $p_n \in \Pi_n$.

   (c) Show that the optimal polynomial approximation p_n^* converges to f in the least-squares sense, that is, $E[p_n^*] \to 0$ as $n \to \infty$.