Part II

1. Consider the finite difference approximation for the eikonal equation $u_t = |u_x|$:

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x}\max(u_{j+1}^n - u_j^n, u_{j-1}^n - u_j^n, 0).$$

   (a) Show that this is a consistent scheme.

   (b) Show that the scheme can be written as

$$u_j^{n+1} = (1-\lambda)u_j^n + \lambda\max(u_{j+1}^n, u_{j-1}^n, u_j^n),\ \ \lambda = \frac{\Delta t}{\Delta x}.$$

   (c) Show that for suitable values of λ, this scheme is stable in the $||\cdot||_{\ell^\infty}$ norm in the spatial domain.

   (d) Let $Z_j^n = u_j^n - u(x_j, t^n)$, where $u(x,t)$ a smooth solution of the PDE. Show that

$$|Z_j^{n+1}| \le \max_j(|Z_{j+1}^n|, |Z_j^n|, |Z_{j-1}^n|) + \text{Const}\,\Delta t\Delta x,$$

   and conclude that

$$|u_j^n - u(x_j, t^n)| \le \text{Const}\,T\Delta x,$$

   where $T = n\Delta t$. (Hint: verify the inequality $|\max_j X_j - \max_j Y_j| \le \max_j |X_j - Y_j|$.)

2. Consider solving the following differential equation with periodic boundary conditons:

$$u_t + au_x = \epsilon u_{xx},\ \ t > 0, 0 \le x < 1$$

   where a and ϵ are two positive constants, and

$$u(x,0) = f(x).$$

   (a) Set up a uniform grid over the interval $[0,1)$ and using the PDE, derive a system of ODEs for $\{\hat{u}_k\}$, where

$$u(x_j, t) = \sum_{k=0}^{N-1} \hat{u}_k(t)e^{-2\pi i k x_j},\ \ x_j = j/N, j = 0, \cdots, N-1.$$

   (b) Propose a stable ODE scheme for solving the systems $\hat{u}_k(t)$. Justify your answer.

   (c) What is the computational complexity of solving the system in part (b) and use it for computing approximation of $u(x,t)$ in the time interval $0 < t \le 1$.

3. Consider the following heat equation in two space dimensions,

$$\begin{aligned} u_t &= \Delta u, \quad |x| < 1, |y| < 1, t > 0, \\ u &= 0, \quad |x| \le 1, |y| = 1, t > 0, \\ u_x &= u, \quad x = -1, |y| < 1, t > 0, \\ u_x &= -u, \quad x = 1, |y| < 1, t > 0, \\ u(x,y,0) &= u_0(x,y), \ \ |x| \le 1, |y| \le 1. \end{aligned}$$

(a) Derive a numerical approximation $u_h^n(x,y)$ by rewriting the heat equation in a variational form and formulating a method based on finite elements in space and Crank-Nicolson (trapezoidal rule) in time, $t_n = n\Delta t, n = 0, 1, \cdots$.

(b) Prove L^2-stability: $||u_h^n||_{L^2(|x|<1,|y|<1)} \le ||u_0||_{L^2(|x|<1,|y|<1)}$.