THE UNIVERSITY OF TEXAS AT AUSTIN
DEPARTMENT OF MATHEMATICS

Preliminary Examination in Probability
Part I

August 19th, 2013

Problem 2.1. Let $X_n \geq 0$ be a sequence of random variables such that there exist $0 < \alpha < \beta$ with the property that

$$\mathbb{E}[X_n^\alpha] \to 1, \quad \mathbb{E}[X_n^\beta] \to 1$$

for $n \to \infty$. Show that $X_n \to 1$ in probability.

Problem 2.2. Let $X_0 = (1, 0)$ and define $X_n \in \mathbb{R}^2$ inductively by declaring that X_{n+1} is chosen at random from the ball of radius $|X_n|$ centered at the origin, i.e.. $X_{n+1}/|X_n|$ is uniformly distributed on the ball of radius one and indepedent of $X_1, \ldots, X_n$. Prove that

$$\frac{\log(|X_n|)}{n} \to c, \quad a.s.$$

and compute c.

Problem 2.3.

(1) Consider an adapted process $(X_n)_n$ such that for any bounded stopping time T we have $X_T \in L^1$ and

$$\mathbb{E}[X_T] = \mathbb{E}[X_0].$$

Show that X is a martingale.

(2) Let $(X_n)_n$ a simple but non-symmetric random walk with probabilities to go up and down p and q. Fix a time horizon N (deterministic). Compute

$$\sup_{0 \leq T \leq N} \mathbb{E}[X_T]$$

where supremum is taken over all possible stopping times.