PRELIMINARY EXAM: ALGEBRAIC TOPOLOGY

Date: August 2013.
Instructions: Do all three problems.
Time Limit: 90 minutes.

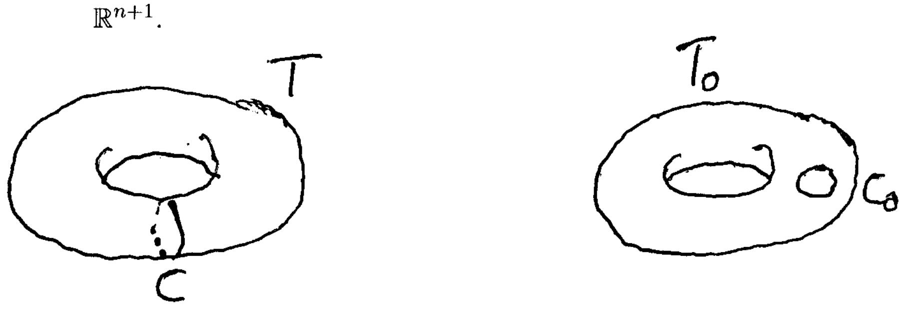
Problem 1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and let s_0 be the point $(1, 0, \ldots, 0) \in S^n$. Define $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $f : (S^n, s_0) \to (X, x_0)$. Let $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering projection. Show that for $n \geq 2$ the map $p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ defined by $p_*([f]) = [p \circ f]$ is a bijection.

Problem 2. Let T be a 2-torus, and T_0 a 2-torus with the interior of a disk removed. Let $C \subset T$ be the simple closed curve illustrated below and $C_0 \subset T_0$ be the boundary of the removed disk. Let X be the space obtained by gluing T and T_0 together via a homeomorphism $h : C \to C_0$.

(a) Compute the homology groups $H_q(X)$.
(b) X has a 2-fold covering space $\widetilde{X}$ with $\dim H_1(\widetilde{X}, \mathbb{Z}_2) = 5$. What is $\dim H_2(\widetilde{X}, \mathbb{Z}_2)$?

Problem 3.

(a) Define the *degree* $deg(f)$ of a map $f : S^n \to S^n$.
(b) Let $f, g : S^n \to S^n$ be maps such that $|deg(f)| \neq |deg(g)|$. Show that there exists $x \in S^n$ such that $f(x)$ and $g(x)$ are orthogonal in $\mathbb{R}^{n+1}$.



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