PRELIMINARY EXAM: DIFFERENTIAL TOPOLOGY

Date: August 2013

Instructions: Do all three problems.

Time Limit: 90 minutes.

Problem 1. Given disjoint oriented manifolds $M, N \subset \mathbb{R}^{k+1}$, the *linking map* $\lambda : M \times N \to S^k$ is given by

$$\lambda(x, y) = \frac{x - y}{|x - y|}.$$

If $dim(M) + dim(N) = k$, the *linking number* $l(M, N)$ of M and N is defined to be the degree of the linking map. Show that if M bounds an oriented manifold with boundary $W \subset \mathbb{R}^{k+1}$ which is disjoint from N, then $l(M, N) = 0$.

Problem 2. Assume that $n \geq 1$.

(a) Prove or disprove: Every smooth compact manifold M^n admits a smooth vector field v with only finitely many zeros. (Hint: you may assume the manifold embeds in $\mathbb{R}^N$ for some N.)

(b) Prove or disprove: Every smooth compact manifold M^n admits a smooth vector field v with no zeros.

(c) Recall that a Lie group is a smooth manifold G endowed with a group structure such that the multiplication map $G \times G \to G$ and the inverse map $G \to G$ are both smooth. If G is a compact Lie group, compute its Euler characteristic $\chi(G)$ and prove that your answer is correct.

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Problem 3.

(a) Let v, w be vectors in $\mathbb{R}^3$, and let v', w' be the (orthogonal) projections of those vectors onto the plane P given by the equation $x + 2y + 2z = 0$. Write down a 2-covector α_0 such that $\alpha_0(v, w)$ is the (signed) area of the parallelogram spanned by v' and w'. (Your answer should depend on your choice of orientation for the plane. State your choices clearly.)

(b) Write down a 2-form α on $\mathbb{R}^3$ such that the integral of α over a compact, oriented surface-with-boundary S gives the signed area of the orthogonal projection of S onto P.

(c) Compute the integral of α over the northern hemisphere of the unit sphere, oriented via the outward normal.