Algebra Prelim Part B January 2015

1. Describe the subgroup of S_n that fixes the polynomial x_1+x_2 under the standard action on $\mathbf{Q}[x_1,\ldots,x_n]$ obtained by permuting variables. Use this to give an upper bound for the degree over $\mathbf{Q}$ of the real part of $z \in \mathbf{C}$, if z is algebraic of degree n over $\mathbf{Q}$. Give a condition under which the bound is sharp. (Here $\mathbf{Q}, \mathbf{C}$ are the fields of rational and complex numbers, respectively).

2. Let K be a field and $f(x) \in K[x]$ be a separable, irreducible polynomial of degree 5. If a, b are distinct roots of $f(x)$ with $K(a) = K(b)$, show that $K(a)/K$ is Galois.

3. Let $K/\mathbf{Q}$ be an extension of degree n, where $\mathbf{Q}$ is the field of rational numbers. Show that the number of subfields of K is at most $2^{n!}$. Suppose that $K = \mathbf{Q}(\alpha, \beta)$ and prove that there exists $m, 0 \leq m \leq 2^{n!}$ such that $K = \mathbf{Q}(\alpha + m\beta)$.