Algebra Prelim part B

~~January 7, 2014~~ 2015 Aug

Directions: You have 90 minutes. Solve two of the three problems. Clearly mark which ones you want graded.

B1. Let l and p be primes. Show that the number of irreducible monic polynomials over $\mathbf{F}_p$, of degree l, is equal to $(p^l - p)/l$.

B2. Suppose k is an algebraically closed field, V is a finite-dimensional vector space over k, and $M : V \to V$ is a linear transformation. Show that there exists a unique pair of linear transformations $D, N : V \to V$ with the following properties. (For existence you can use well-known results, but for uniqueness you should argue directly.)

(1) $M = N + D$.

(2) N is nilpotent, i.e. $N^s = 0$ for some integer $s > 0$.

(3) D is diagonalizable, i.e. V has a basis of D-eigenvectors.

(4) Every linear transformation G commuting with M also commutes with N and D.

B3. Let E be the splitting field of $x^7 - 3$ over $\mathbf{Q}$.

(a) Determine the Galois group $\mathrm{Gal}(E/\mathbf{Q})$ as a group of permutations of the roots of $x^7 - 3$.

(b) Find a primitive generator of $E/\mathbf{Q}$.

(c) Prove that E is not a subfield of any cyclotomic extension of $\mathbf{Q}$.

(d) Describe all the subfields of $E/\mathbf{Q}$ that are Galois over $\mathbf{Q}$.