PRELIMINARY EXAMINATION IN ANALYSIS
PART II - COMPLEX ANALYSIS
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Please solve all four of the following problems.

(1) Let U denote a bounded simply connected region in the complex plane and let $f : U \to U$ be an analytic function such that $f(0) = 0$, and $|f'(0)| < 1$. Let

$$f^{(n)}(z) = f \circ f \circ \cdots \circ f(z)$$

be the function obtained by composing f n-times. Prove that $f^{(n)}(z) \to 0$ uniformly on compact subsets of U.

(2) Let f be a non-constant meromorphic function on the complex plane $\mathbb{C}$ that obeys

$$f(z) = f(z + \sqrt{2}) = f(z + i\sqrt{2}).$$

Assume f has at most one pole in the closed disk $D = \{z : |z|1\}$.

(a) Show that f has exactly one pole in D.

(b) Show that this pole is not simple.

(3) Let f be an entire non-constant function such that $f(1 - z) = 1 - f(z)$ for all $z \in \mathbb{C}$. Show that $f(\mathbb{C}) = \mathbb{C}$.

(4) Let $\mathbb{D}$ be the open unit disc in the complex plane and μ the Lebesgue measure on $\mathbb{D}$. Let H be the subspace of $L^2(\mathbb{D}, \mu)$ consisting of holomorphic functions. Show that H is closed in $L^2(\mathbb{D}, \mu)$. Hint: prove that for every compact $K \subset \mathbb{D}$ there exists a constant $C_K > 0$ such that $\|f|_K\|_\infty \leq C_K \|f|_K\|_2$.