PRELIMINARY EXAMINATION IN ANALYSIS
Part I, Real Analysis
January 10, 2017

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1. For $f \in \mathrm{L}^1(\mathbb{R})$ denote by Mf be the restricted maximal function defined by

$$(Mf)(x) = \sup_{0<t<1} \frac{1}{2t} \int_{x-t}^{x+t} \big|f(z)\big| \, dz \,.$$

Show that $M(f * g) \leq (Mf) * Mg$ for all $f, g \in \mathrm{L}^1(\mathbb{R})$.

2. Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Show that if $f \in \mathrm{L}^p(\mathbb{R}^n)$ and $g \in \mathrm{L}^q(\mathbb{R}^n)$ then $f * g$ is bounded and continuous on $\mathbb{R}^n$.

3. Let B be the closed unit ball in $\mathbb{R}^n$, and let $f_1, f_2, f_3, \ldots$ be nonnegative integrable functions on B. Assume that

(i) $f_k \to f$ almost everywhere.

(ii) For every $\varepsilon > 0$ there exists $M > 0$ such that

$$\int_{\{x \in B : f_k(x) > M\}} f_k(x) \, dx < \varepsilon \,, \qquad k = 1, 2, 3, \ldots$$

Show that $f_k \to f$ in $\mathrm{L}^1(B)$.

4. Let $f, f_1, f_2, \ldots$ be increasing functions on $[a, b]$. If $\sum_k f_k$ converges pointwise to f on $[a, b]$, show that $\sum_k f_k'$ converges to f' almost everywhere on $[a, b]$.