1. Let $X$ be a topological space. Recall that the suspension $SX$ of $X$ is obtained from $X \times [-1, 1]$ by collapsing $X \times \{1\}$ to a point and $X \times \{-1\}$ to another point.

(a): Show that if $X$ is path connected then $SX$ is simply connected.

For parts (b) and (c) you may work in either singular homology or simplicial homology. If you chose to work in simplicial homology, you may assume $X$ is a simplicial complex (or ∆-complex).

(b): Calculate the homology groups of $SX$ in terms of the homology groups of $X$.

(c): Prove that if $SX$ retracts onto $X \times \{0\}$ then the reduced homology groups $\tilde{H}_k(X)$ are all zero.

2. Let $X = M_2$ be the closed orientable surface of genus 2 shown in the figure below. Let $\gamma = a \overline{c}$ be the concatenated loop that traverses $a$ and then $c$ backwards. Note that $\gamma$ crosses itself once at the base point. Show that there exists a two-fold covering space $p: \tilde{X} \to X$ for which $\gamma$ lifts to an embedded circle (i.e. a loop that does not cross itself). Do this in two steps:

(a): First, translate the problem into a problem about group theory.

(b): Solve the group theory problem from (a).

3. Recall that the local homology groups of a space $X$ at a point $x \in X$ are defined to be the relative homology groups $H_k(X, X - \{x\})$.

(a): Calculate the local homology groups for Euclidean space $\mathbb{R}^m$.

(b): Prove the Invariance of Dimension theorem: If non-empty open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

4. Consider the unit cube $C = [-1, 1] \times [-1, 1] \times [-1, 1]$ with its usual CW structure. Let $X$ be the union of the 2-skeleton of $C$ (i.e. the six square faces of $C$) together with the three line segments passing through the origin connecting midpoints of opposite faces, $\{0\} \times \{0\} \times [-1, 1]$, $\{0\} \times [-1, 1] \times \{0\}$, and $[-1, 1] \times \{0\} \times \{0\}$. Let $Y$ be the 1-skeleton of $C$ (i.e. the twelve edges of $C$).

(a): Compute $\pi_1(X)$ and $\pi_1(Y)$ and show they are isomorphic.

(b): Does $X$ retract onto $Y$?