Preliminary Examination: Algebraic topology. January 10, 2018

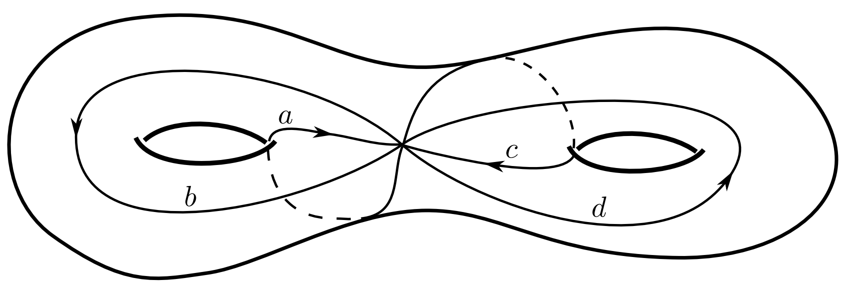
Instructions: Do any **three** of the four questions. (Only turn in three).

Time limit: 90 minutes.

1. Let X be a topological space. Recall that the suspension SX of X is obtained from $X \times [-1, 1]$ by collapsing $X \times \{1\}$ to a point and $X \times \{-1\}$ to another point.
(a): Show that if X is path connected then SX is simply connected.

For parts (b) and (c) you may work in either singular homology or simplicial homology. If you chose to work in simplicial homology, you may assume X is a simplicial complex (or Δ-complex).
(b): Calculate the homology groups of SX in terms of the homology groups of X.
(c): Prove that if SX retracts onto $X \times \{0\}$ then the reduced homology groups $\widetilde{H}_k(X)$ are all zero.

2. Let $X = M_2$ be the closed orientable surface of genus 2 shown in the figure below. Let $\gamma = a\bar{c}$ be the concatenated loop that traverses a and then c backwards. Note that γ crosses itself once at the base point. Show that there exists a two-fold covering space $p : \widetilde{X} \to X$ for which γ lifts to an emdedded circle (i.e. a loop that does not cross itself). Do this in two steps:
(a): First, translate the problem into a problem about group theory.
(b): Solve the group theory problem from (a).



3. Recall that the local homology groups of a space X at a point $x \in X$ are defined to be the relative homology groups $H_k(X, X - \{x\})$.
(a): Calculate the local homology groups for Euclidean space $\mathbb{R}^m$.
(b): Prove the *Invariance of Dimension* theorem: If non-empty open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are homeomorphic, then $m = n$.

4. Consider the unit cube $C = [-1, 1] \times [-1, 1] \times [-1, 1]$ with its usual CW structure. Let X be the union of the 2-skeleton of C (i.e. the six square faces of C) together with the three line segments passing through the origin connecting midpoints of opposite faces, $\{0\} \times \{0\} \times [-1, 1]$, $\{0\} \times [-1, 1] \times \{0\}$, and $[-1, 1] \times \{0\} \times \{0\}$. Let Y be the 1-skeleton of C (i.e. the twelve edges of C).
(a): Compute $\pi_1(X)$ and $\pi_1(Y)$ and show they are isomorphic.
(b): Does X retract onto Y?