Work 4 of the following 5 problems.

1. Given $z \in \mathbb{C}$ and a smooth closed curve $\gamma$ in $\mathbb{C} \setminus \{z\}$, denote by $n(\gamma, z)$ the index or winding number of $\gamma$ about $z$. If $\gamma$ admits a representation

$$\gamma(t) = \sum_{k=-N}^{N} c_k e^{ikt}, \quad t \in [0, 2\pi],$$

with $c_{-N}$ and $c_N$ not both zero, show that $-N \leq n(\gamma, z) \leq N$.

2. Evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 - x} \, dx$. Explain all steps carefully.

3. Assume that $f$ is meromorphic on $\mathbb{C}$ and bounded outside some compact set. Show that $f$ is a rational function.

4. Prove that

$$\frac{d}{dz} \left( \frac{\pi \sin(z)}{\sin(\pi z)} \right) = \sum_{n \in \mathbb{Z}} \frac{(-1)^{n+1} \sin(n)}{(z - n)^2}$$

for all $z \in \mathbb{C} \setminus \mathbb{Z}$, with the right-hand side converging uniformly on every compact subset of $\mathbb{C} \setminus \mathbb{Z}$.

5. Determine all analytic functions $f : H \to \mathbb{C}$ on the half-plane $H = \{z \in \mathbb{C} : \text{Re} \, z > 0\}$ that satisfy $f(\sqrt{n}) = n$ and $|f^{(n)}(1)| \leq 3$ for all positive integers $n$. 