1. Let \((H, \langle \cdot, \cdot \rangle)\) be a complex Hilbert space.
   (a) Prove that a linear operator \(T : H \rightarrow H\) is self adjoint if and only if \(\langle Tx, x \rangle\) is real for all \(x \in H\).
   (b) State the Spectral Theorem for a compact, self-adjoint operator.
   (c) Let \(T\) be a compact, self-adjoint operator. Prove that there are positive operators \(P\) and \(N\) such that \(T = P - N\) and \(PN = 0\).

2. Let \(X\) and \(Y\) be separable and reflexive Banach spaces.
   (a) Show that if \(x_n \xrightarrow{w} x\) in \(X\), then \(\liminf_{n \to \infty} \|x_n\| \geq \|x\|\).
   (b) State the Banach-Alaoglu Theorem.
   (c) Suppose that \(D \subset Y\) is a dense subset. Let \(T : X \rightarrow Y\) be a bounded linear map that is bounded below. Suppose further that, given any \(g \in D\), \(Tx = g\) is uniquely solvable for \(x \in X\). Prove that there is a unique solution \(x \in X\) to \(Tx = f\) for any \(f \in Y\).

3. Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space of real or complex functions defined on \(\mathbb{R}^d\). A reproducing kernel function for \(H\) is a function \(K(\cdot, \cdot)\) defined on \(\mathbb{R}^d \times \mathbb{R}^d\) that satisfies
   i. \(K(\cdot, y) \in H\) for all \(y \in \mathbb{R}^d\) and
   ii. \(f(y) = \langle f, K(\cdot, y) \rangle\) for all \(f \in H\) and \(y \in \mathbb{R}^d\).
   (a) Prove that \(H\) has a reproducing kernel function if and only if for each \(y \in \mathbb{R}^d\), the linear functional \(L_y\) defined by \(L_y(f) = f(y)\) is bounded.
   (b) If \(K\) exists, show that \(K\) is unique and that \(K(x, y) = \overline{K(y, x)}\).
   (c) If \(K\) exists, \(L \in H^*\), and \(z(y) = LK(\cdot, y)\), show that \(z \in H\) and \(\|L\|^2 = Lz\).
1. Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions on $\mathbb{R}$. Under what conditions on the complex sequence $\{a_k\}_{k=1}^{\infty}$ is $\sum_{k=1}^{\infty} a_k \delta_k \in \mathcal{S}'(\mathbb{R})$? Here, $\delta_k$ is the point mass centered at $x = k$.

2. Show the following two statements about Sobolev spaces, where $\Omega \subset \mathbb{R}^d$ is a domain.

   (a) There is no embedding of $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq p \leq d$ and $q > dp/(d - p)$. [Hint: Show a counterexample with the function $f(x) = |x|^\alpha$ by choosing an appropriate domain $\Omega$ and exponent $\alpha$.]

   (b) There is no embedding of $W^{1,p}(\Omega) \hookrightarrow C^0_B(\Omega)$ for $1 \leq p < d$. Note that in the previous case, $f$ is not bounded. What can you say about which (negative) Sobolev spaces the Dirac mass lies in?

3. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary and $\{u_k\} \subset H^{2+\varepsilon}(\Omega)$ is a bounded sequence, where $\varepsilon > 0$.

   (a) Show that there is $u \in H^2(\Omega)$ such that, for a subsequence, $\{u_{k_j}\}_{k=1}^{\infty} \to u$ in $H^2(\Omega)$.

   (b) Find all $q$ and $s \geq 0$ such that, for a subsequence, $\{u_{k_j}\} \to u$ in $W^{s,q}(\Omega)$. 