Preliminary Examination: Algebraic topology. January 16, 2019

Instructions: Answer all three questions

Time limit: 90 minutes.

1. Let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disk and let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote its boundary, the unit circle. Let p, q, r, s be integers so that $ps - qr = 1$. Then the map $f : S^1 \times S^1 \to S^1 \times S^1$ defined by

$$f(e^{i\theta_1}, e^{i\theta_2}) = (e^{i(p\theta_1 + q\theta_2)}, e^{i(r\theta_1 + s\theta_2)})$$

is a homeomorphism which takes the basepoint $(1, 1)$ to itself.

(a). Write down $\pi_1(S^1 \times S^1, (1, 1))$ and give an explicit generating set.

(b). Compute the induced map $f_* : \pi_1(S^1 \times S^1, (1, 1)) \to \pi_1(S^1 \times S^1, (1, 1))$.

For parts (c),(d), let X be the space obtained by gluing a solid torus $S^1 \times D^2$ to another solid torus $S^1 \times D^2$ along their boundary tori using the homeomorphism f.

(c). Compute the fundamental group of X.

(d). Use the Mayer-Vietoris sequence to calculate the homology groups $H_k(X)$ for all k.

2. Let $X = S^1 \vee S^1$ be the wedge of two circles, a let $x \in X$ be the point where the two circles meet (the "wedge" point). Then $\pi_1(X, x) \cong F_2 = \langle a, b \rangle$ is the free group on two generators.

(a) Give an example of a normal subgroup $H \lhd F_2$ with index $[F_2 : H] = 4$. Draw the associated covering space.

(b) Give an example of a subgroup $H < F_2$ with index $[F_2 : H] = 4$ for which $N(H) = H$, where $N(H) = \{g \in F_2 : gHg^{-1} = H\}$ denotes the normalizer of H. Draw the associated covering space.

3. In this problem, M_g denotes the closed orientable surface of genus g.

(a) Write down, without proof, the homology groups $H_k(M_g)$ for all k, and the Euler characteristic $\chi(M_g)$.

(b) Suppose $p : M_h \to M_g$ is a finite covering space, where here $g \geq 2$. Use Euler characteristic to give a formula for h in terms of g and the degree of the cover.

For parts (c),(d), assume that $g > h \geq 2$, let $x_0 \in M_h$ be a basepoint, and let $f : M_h \to M_g$ be any continuous map.

(c) Show that the image of $f_* : \pi_1(M_h, x_0) \to \pi_1(M_g, f(x_0))$ has infinite index.

(d) Show that $f_* : H_2(M_h) \to H_2(M_g)$ is the zero map. You may use the following fact:

$(*)$ If $\widetilde{M}$ is a *non-compact* surface, then $H_2(\widetilde{M}) = 0$.