Differential Topology Prelim  
Wednesday, January 16, 2019, 2:40 – 4:15 PM

Instructions: Do all three problems. All manifolds are assumed to be smooth. If you have any questions, ask!

1. Let $X^n$ be an $n$-manifold, $Y^m$ an $m$-manifold, and $f : X^n \to Y^m$ a smooth map.
   a) Show that if $p$ is a regular value of $f$, then $f^{-1}(p)$ is a codimension-$m$ submanifold of $X$. You may use without proof the inverse function theorem and its corollaries.
   b) If $f^{-1}(p)$ is a codimension-$m$ submanifold of $X$, does that imply that $p$ is a regular value? Either prove it is or give a counterexample.
   c) Let $g : X^n \to \mathbb{R}^5$ be a smooth map. Show that there exists a 2-plane $P \subset \mathbb{R}^5$ such that $g^{-1}(P)$ is a codimension-3 submanifold of $X$.

2a) Show that $\mathbb{CP}^1$ is a smooth 2-manifold.
   b) Writing $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, let $f_0 : \mathbb{CP}^1 \to \mathbb{CP}^1$ be defined by
      $$f_0(z) = \begin{cases} 
      z^3 & z \in \mathbb{C} \\
      \infty & z = \infty
      \end{cases}.$$ 
      Show that $f_0$ is a smooth map. (This is trivial everywhere except around $z = \infty$, so you only need to write down your argument for what happens around $z = \infty$.)
   c) Homotope $f_0$ to a map $f_1$ that has $\infty$ as a regular value. Then compute the degree of $f_0$ by counting (with sign) the points of $f_1^{-1}(\infty)$.

3a) On $X = \mathbb{R}^3 - \{0\}$, consider the differential form
      $$\alpha = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{4\pi(x^2 + y^2 + z^2)^{3/2}}.$$ 
      Show that $\alpha$ is closed but not exact.
   b) Let $Y$ and $Z$ be isotopic embeddings of a genus-$g$ oriented surface $M_g$ in $X$. That is, there is a smooth map $I : [0, 1] \times M_g \to X$ such that $i_0$ is the embedding of $Y$, $i_1$ is the embedding of $Z$, and each $i_t$ is an embedding of $M_g$ in $X$. Give $Y$ and $Z$ orientations inherited from $M_g$. Show that $\int_Y \alpha = \int_Z \alpha$.
   c) You can take as a given the fact that $Y$ divides $\mathbb{R}^3$ into two pieces, namely an inside and an outside. Show that $\int_Y \alpha = \pm 1$ if the origin is on the inside, and that $\int_Y \alpha = 0$ if the origin is on the outside.