Planar fronts in bistable coupled map lattices

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R. Coutinho, B. Fernandez, *Fronts and interfaces in bistable extended mappings*, Nonlinearity **11** (1998) 1407–1433.

R. Coutinho, B. Fernandez, *Fronts in extended systems of bistable maps coupled by convolutions*, Nonlinearity **17** (2004) 23–47.

The phase space is $[0,1]^{\mathbb{Z}^d}$ and the map (CML) is

$$u_{\mathbf{n}}^{t+1} = \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}}\, f(u_{\mathbf{n}-\mathbf{s}}^t)$$

For each site $\mathbf{n}$ in the lattice $\mathbb{Z}^d$ and for each discrete time $t \in \mathbb{Z}$ the variable $u_{\mathbf{n}}^t \in [0,1]$. The local map is a function $f : [0,1] \to [0,1]$. The convex diffusive coupling is given by the coefficients $\ell_{\mathbf{s}}$ satisfying

$$\ell_{\mathbf{s}} \geqslant 0 \quad \text{and} \quad \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}} = 1.$$

Example: for $d = 2$, $\ell_{0,0} = 1 - \varepsilon$, $\ell_{1,0} = \ell_{-1,0} = \ell_{0,1} = \ell_{0,-1} = \frac{\varepsilon}{4}$ and $\ell_{m,n} = 0$ if $|m| + |n| > 1$.

$$u_{m,n}^{t+1} = (1-\varepsilon)\, f(u_{m,n}^t) + \frac{\varepsilon}{4}\left(f(u_{m+1,n}^t) + f(u_{m,n+1}^t) + f(u_{m-1,n}^t) + f(u_{m,n-1}^t)\right)$$

that is

$$u^{t+1} = f(u^t) + \frac{\varepsilon}{4}\Delta f(u^t)$$

where $\varepsilon \in [0,1]$ is the coupling parameter.

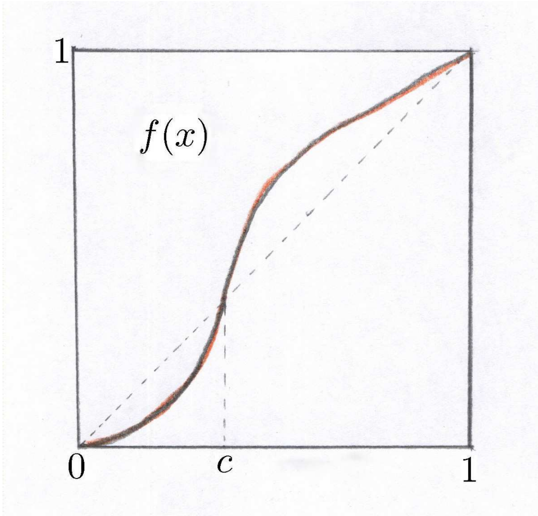
bistable map on $[0,1]$:

a continuous increasing map $f : [0,1] \to [0,1]$

such that there exists $c \in (0,1)$ so that

$$f(x) < x \text{ for all } x \in (0,c) \qquad \text{and} \qquad x < f(x) \text{ for all } x \in (c,1).$$

Two stable fixed points: $f(0) = 0$ and $f(1) = 1$; one unstable fixed point: $f(c) = c$.



Planar Fronts: solutions $u^t_{\mathbf{n}}$ of the form

$$u^t_{\mathbf{n}} = \phi(\mathbf{n} \cdot \mathbf{k} - t\,v) \qquad \forall t \in \mathbb{Z}$$

Where:

$\phi : \mathbb{R} \to [0,1]$ is the **front shape** satisfying $\lim_{x\to-\infty} \phi(x) = 0$ and $\lim_{x\to+\infty} \phi(x) = 1$

k is the direction of propagation ($\mathbf{k} \in \mathbb{R}^d$, $\|\mathbf{k}\| = 1$; i.e. $\mathbf{k} \in \mathbb{S}^{d-1}$)

v is the front velocity

For each direction $\mathbf{k} \in \mathbb{S}^{d-1}$ define the set

$$Z(\mathbf{k}) = \{\omega \in \mathbb{R} : \exists \mathbf{s} \in \mathbb{Z}^d \quad \omega = \mathbf{s} \cdot \mathbf{k}\}.$$

and consider the invariant subset $\mathcal{X}_{\mathbf{k}} \subset [0,1]^{\mathbb{Z}^d}$, defined by

$$\mathcal{X}_{\mathbf{k}} = \left\{ \{u_{\mathbf{n}}\}_{\mathbf{n} \in \mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d} : \mathbf{m} \cdot \mathbf{k} = \mathbf{n} \cdot \mathbf{k} \Rightarrow u_{\mathbf{m}} = u_{\mathbf{n}} \right\}$$

For each $\{u_{\mathbf{s}}\}_{\mathbf{s} \in \mathbb{Z}^d} \in \mathcal{X}_{\mathbf{k}}$ define the function $\psi \in [0,1]^{\mathbb{R}}$, in $Z(\mathbf{k})$ by

$$\psi(\mathbf{s} \cdot \mathbf{k}) = u_{\mathbf{s}}$$

and for $x \in \mathbb{R} \setminus Z(\mathbf{k})$ by $\psi(x) = 0$ (0 being a fixed point of f).

Then the dynamics of the (CML) implies that

$$\psi^{t+1}(x) = \sum_{\mathbf{s} \in \mathbb{Z}^d} \ell_{\mathbf{s}} \, f \circ \psi^t(x - \mathbf{s} \cdot \mathbf{k}).$$

Hence the (CML) dynamics induces a map $F : [0,1]^{\mathbb{R}} \to [0,1]^{\mathbb{R}}$. $\qquad \psi^{t+1} = F(\psi^t)$.

■ If the direction $\mathbf{k}$ is totally irrational ($\forall \mathbf{s} \in \mathbb{Z}^d \quad \mathbf{s} \cdot \mathbf{k} = 0 \Rightarrow \mathbf{s} = \mathbf{o}$), then the (CML) dynamics is a restriction of the dynamics given by the map F, because $\mathcal{X}_{\mathbf{k}} = [0,1]^{\mathbb{Z}^d}$ in this case.

■ If the direction $\mathbf{k}$ is not totally irrational, then the map F is an extension of the (CML) dynamics restricted to the set $\mathcal{X}_{\mathbf{k}}$.

If we restrict the map $F(\psi)(x) = \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}}\, f \circ \psi(x - \mathbf{s} \cdot \mathbf{k})$ to the set $\mathcal{B}$ of Borel measurable functions on $\mathbb{R}$ with values in $[0,1]$, then the map F can be written as

$$F(\psi) = h_{\mathbf{k}} * f \circ \psi,$$

where the convolution is defined by the Lebesgue-Stieltjes integral

$$h * \varphi(x) = \int_{\mathbb{R}} \varphi(x - y)\; dh(y)$$

and the distribution function $h_{\mathbf{k}} : \mathbb{R} \to [0,1]$ is $\quad h_{\mathbf{k}}(x) = \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}}\, H(x - \mathbf{s} \cdot \mathbf{k})$

and H is the Heaviside function: $\quad H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$

Reformulating the problem:

Given a distribution function $h : \mathbb{R} \to [0,1]$ (increasing function with the following limits $\lim_{x\to-\infty} h(x) = 0$ and $\lim_{x\to+\infty} h(x) = 1$) and a bistable map $f : [0,1] \to [0,1]$ define the map $F : \mathcal{B} \to \mathcal{B}$ by

$$F(\psi) = h * f \circ \psi$$

Find a front shape $\phi \in \mathcal{B}$ and a velocity v, satisfying:

$$T^v \phi = F(\phi) \quad \text{, with} \quad \lim_{x\to-\infty} \phi(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} \phi(x) = 1$$

where T^v is the **translation** by $v \in \mathbb{R}$ defined by $\quad T^v u(x) = u(x - v)$

Results

Existence of fronts

Theorem *For any distribution function* h *and* ***any bistable map*** f*, there exists a velocity* $v \in \mathbb{R}$ *and an increasing function* ϕ *such that*

$$\lim_{x\to-\infty} \phi(x) = 0, \quad \lim_{x\to+\infty} \phi(x) = 1 \quad \textit{and} \quad T^v\phi = F\phi,$$

where $F\phi = h * f \circ \phi$.

Uniqueness of front velocity

Theorem *For any distribution function* h *and* ***any bistable regular map*** f*, there exists a* ***unique*** *velocity* $v \in \mathbb{R}$ *and an increasing function* ϕ *such that*

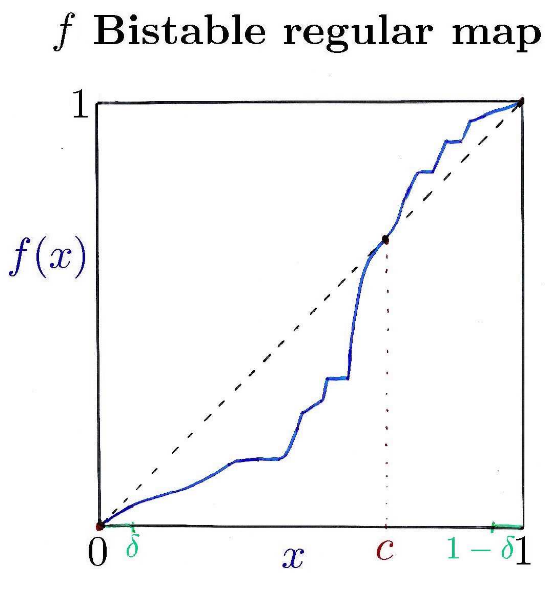
$$\lim_{x\to-\infty} \phi(x) = 0, \quad \lim_{x\to+\infty} \phi(x) = 1 \quad \textit{and} \quad T^v\phi = F\phi,$$

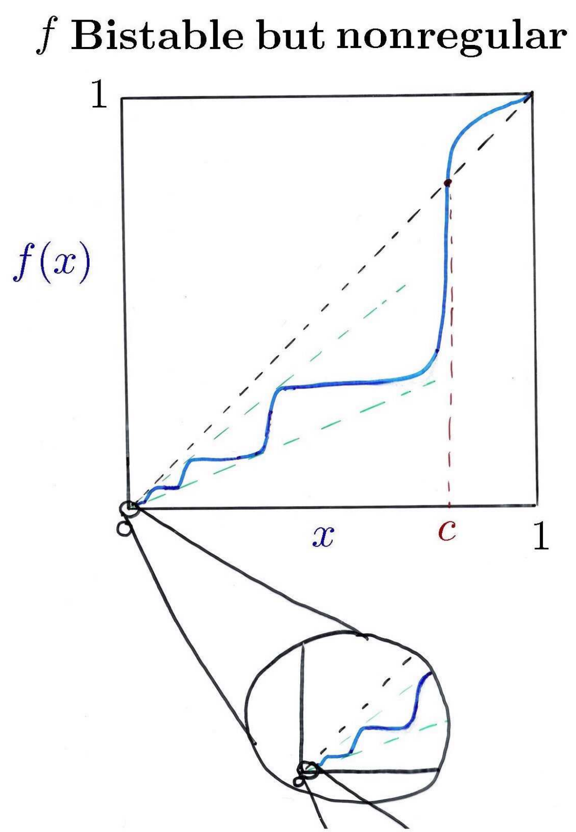
where $F\phi = h * f \circ \phi$.

Bistable regular maps

A bistable map is said to be regular if it is a weak contraction in a neighbourhood of each stable fixed point. That is

$$\exists \delta > 0 \quad \left[x,y \in (0,\delta) \quad \text{or} \quad x,y \in (1-\delta,1) \right] \quad \Rightarrow |f(x) - f(y)| \leqslant |x-y|.$$





Using the Taylor's formula we find the following sufficient conditions for a bistable map f to be **regular**:
a) f analytic or b) $f \in C^1$ and $f'(0) < 1$ and $f'(1) < 1$ or $f \in C^2$ and $f''(0) \neq 0$ and $f''(1) \neq 0$ or d) $f \in C^3$ and $f'''(0) \neq 0$ and $f'''(1) \neq 0$ or ...

Nevertheless it is possible to construct C^∞ bistable maps that are not regular

Continuity of the front velocity

Assume that f is **regular** and let $v(f,h)$ be the unique front velocity of F.

Theorem *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of regular bistable maps which converges pointwise to a bistable regular map f. Let $\{h_n\}_{n\in\mathbb{N}}$ be a sequence of distribution functions and h be a distribution function such that $\lim_{n\to\infty} d(h_n,h) = 0$. Then $\lim_{n\to\infty} v(f_n,h_n) = v(f,h)$.*

Given two right continuous distribution functions h and h', define the **Lévy distance**

$$d(h,h') = \inf\{\varepsilon > 0 \ : \ h(x-\varepsilon) - \varepsilon \leqslant h'(x) \leqslant h(x+\varepsilon) + \varepsilon, \quad \forall x \in \mathbb{R}\}.$$

This is the same as the **Hausdorff distance** restricted to graphs of distribution functions.
Hausdorff distance:

$$d(h,h') = \max\left\{ \sup_{z_1\in Gh} \inf_{z_2\in Gh'} \|z_1 - z_2\|, \sup_{z_1\in Gh'} \inf_{z_2\in Gh} \|z_1 - z_2\| \right\},$$

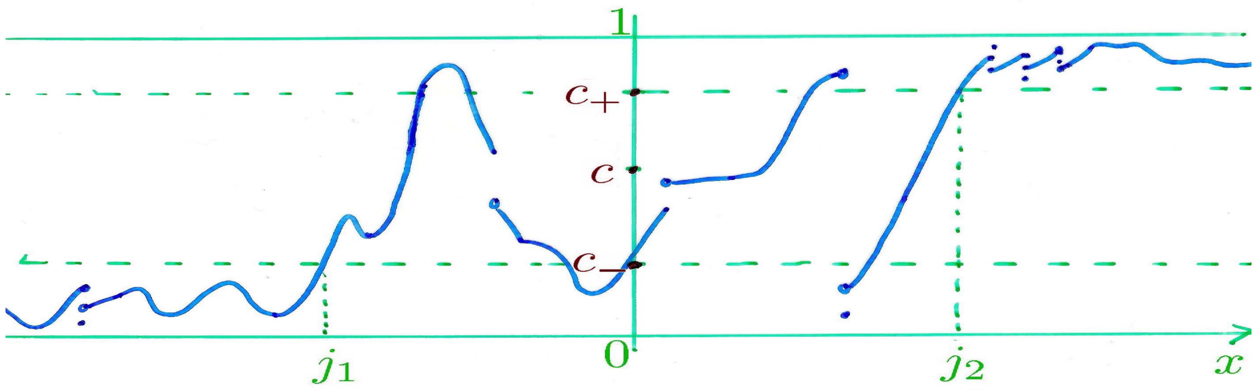
where Gh is the graph of h, i.e. $Gh = \{(x,y) \ : \ h(x^-) \leqslant y \leqslant h(x^+)\}$,
and the $\mathbb{R}^2$ norm $\|.\|$ is given by $\|(x,y)\| = \max\{|x|,|y|\}$

Interfaces and reference centers $J_a(\psi)$

Recall that c denotes the unstable fixed point of f.

An **interface** is a function $u \in \mathcal{B}$ if there exists $c_- \in (0,c)$, $c_+ \in (c,1)$ and $j_1 \leqslant j_2 \in \mathbb{R}$ so that

$$u(x) \leqslant c_- \quad \text{if} \quad x \leqslant j_1 \quad \text{and} \quad u(x) \geqslant c_+ \quad \text{if} \quad x \geqslant j_2.$$



If u is an interface, then the iterate $F^t u$ is also an interface ($t \geqslant 0$).

Moreover for sufficiently large t, $F^t u$ is an interface for c_- arbitrarily near 0 and c_+ arbitrarily near 1.

Given the level $a \in (0,1)$, the **reference center** of a function u is

$$J_a(u) = \inf\{x \in \mathbb{R} \ : \ u(x) \geqslant a\}.$$

By applying a translation a function u can be centered at 0: $J_a(T^{-J_a(u)}u) = 0.$

Velocity of interfaces

Theorem *Let h be a distribution function and let f be a regular bistable map. For every interface u and every $a \in (0,1)$, we have*

$$\lim_{t\to\infty} \frac{J_a(F^t u)}{t} = v(f,h).$$

Where $v(f,h)$ is the (unique) front velocity of F.

Application to the planar fronts of (CML)

$$u_{\mathbf{n}}^{t+1} = \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}}\, f(u_{\mathbf{n}-\mathbf{s}}^t), \quad \text{with } f \text{ bistable}, \ \ell_{\mathbf{s}} \geqslant 0 \text{ and } \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}} = 1.$$

If ϕ is a function such that $\lim_{x\to-\infty} \phi(x) = 0$, $\lim_{x\to+\infty} \phi(x) = 1$ and $T^v\phi = F\phi$, where $F\phi = h_{\mathbf{k}} * f \circ \phi$, then defining

$$u_{\mathbf{n}}^t = \phi(\sigma + \mathbf{n}\cdot\mathbf{k} - tv)$$

(for an arbitrary phase $\sigma \in \mathbb{R}$), we have $u_{\mathbf{n}}^{t+1} = \sum_{\mathbf{s}\in\mathbb{Z}^d} \ell_{\mathbf{s}}\, f(u_{\mathbf{n}-\mathbf{s}}^t)$, i.e. $u_{\mathbf{n}}^t$ is a planar front with direction of propagation $\mathbf{k}$ and front shape ϕ.

Existence of planar fronts

Theorem *For any direction $\mathbf{k} \in \mathbb{S}^{d-1}$ and any bistable function f there exists planar fronts in the direction $\mathbf{k}$ for (CML). If f is regular, then the velocity of these fronts is uniquely determined for each direction $\mathbf{k}$.*

The velocity of fronts v depends then on f, $\{\ell_{\mathbf{s}}\}_{\mathbf{s}\in\mathbb{Z}^d}$ and $\mathbf{k}$:

$$v = v(f, \{\ell_{\mathbf{s}}\}_{\mathbf{s}\in\mathbb{Z}^d}, \mathbf{k})$$

Continuity of the velocity

Theorem *Given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* f, f' *regular bistable,* $\{\ell_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}$, $\{\ell'_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}$ *nonnegative and normalized and* $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{S}^{d-1}$ *satisfying*

$$\|\boldsymbol{k} - \boldsymbol{k}'\| < \delta, \quad \sum_{\boldsymbol{s}\in\mathbb{Z}^d} |\ell'_{\boldsymbol{s}} - \ell_{\boldsymbol{s}}| < \delta \quad \textit{and} \quad \sup|f - f'| < \delta,$$

we have

$$\left|v(f, \{\ell_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}, \boldsymbol{k}) - v(f', \{\ell'_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}, \boldsymbol{k}')\right| < \varepsilon.$$

Interfaces

A configuration $\{u_{\mathbf{s}}\}_{\mathbf{s}\in\mathbb{Z}^d} \in [0,1]^{\mathbb{Z}^d}$ is an interface in the direction $\mathbf{k}$ if there exists j_-, j_+, $0 < c_- < c < c_+ < 1$, such that

$$\mathbf{n}\cdot\mathbf{k} < j_- \Rightarrow u_{\mathbf{n}} < c_- \quad \text{and} \quad \mathbf{n}\cdot\mathbf{k} > j_+ \Rightarrow u_{\mathbf{n}} > c_+.$$

(note that $\mathbf{n}\cdot\mathbf{k}$ measures a position along the line orthogonal to $\mathbf{k}$)

Theorem *If* $\{u^0_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d, t\in\mathbb{N}}$ *is an interface in the direction* $\boldsymbol{k}$*, then the evolution* $\{u^t_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}$ *by (CML) is an interface in the direction* $\boldsymbol{k}$ *and*

$$\lim_{t\to+\infty} \frac{1}{t} J^{\boldsymbol{k}}_a(\{u^t_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}) = v(f, \{\ell_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}, \boldsymbol{k}).$$

Where $J^{\boldsymbol{k}}_a(\{u_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}) = \inf\left\{ j \in \mathbb{R} : \boldsymbol{n}\cdot\boldsymbol{k} > j \Rightarrow u_{\boldsymbol{n}} \geqslant a\right\}$ *and* $a \in (0,1)$.

Extended bistable maps

$$u^{t+1} = Fu^t := h * f \circ u^t$$

The phase space is the set $\mathcal{B}$ of Borel-measurable functions on $\mathbb{R}$ with values in $[0,1]$.

Basic properties:

Homogeneity

$$T^v F = F T^v \quad \text{for all } v \in \mathbb{R}.$$

Continuity:

$$\forall x \in \mathbb{R} \quad \lim_{n\to\infty} u_n(x) = u(x) \qquad \Rightarrow \qquad \forall x \in \mathbb{R} \quad \lim_{n\to\infty} Fu_n(x) = Fu(x).$$

Monotony:

$$u \leqslant v \qquad \Rightarrow \qquad Fu \leqslant Fv.$$

Sketch of the proof of existence of fronts

Let $\mathcal{I} \subset \mathcal{B}$ be the subset composed of increasing functions, $v \in \mathbb{R}$ and $c_+ \in (c, 1)$.
The set $\mathcal{S}_{v,c_+}$ of **sub-fronts** of velocity v:

$$\mathcal{S}_{v,c_+} = \left\{\psi \in \mathcal{I} \ : \ F\psi \leqslant T^v\psi \text{ and } J_{c_+}(\psi) = 0\right\}.$$

When $\mathcal{S}_{v,c_+}$ is not empty, consider the function

$$\eta_v(x) = \inf_{\psi \in \mathcal{S}_{v,c_+}} \psi(x), \quad x \in \mathbb{R}.$$

It turns out that $\eta_v \in \mathcal{S}_{v,c_+}$ and therefore η_v is a **minimal sub-front** of velocity v.

We also prove the existence of a maximal sub-fronts velocity $\bar{v} = \max\{v \in \mathbb{R} \ : \ \mathcal{S}_{v,c_+} \neq \emptyset\}$.

Consider the reference centers of the iterates $F^n\eta_{\bar{v}}$ of the minimal sub-front $\eta_{\bar{v}}$ for the maximal sub-fronts velocity $\bar{v}$:

$$j_n := J_{c_+}(F^n\eta_{\bar{v}})$$

Then we prove that $\lim_{n\to\infty} \inf(j_{n+m} - j_n) = m\bar{v}$.

From this we use an arithmetical Lemma that ensures that there exists a strictly increasing sequence $\{n_k\}$ such that for all m

$$\lim_{k\to\infty}(j_{n_k+m} - j_{n_k}) = m\bar{v}.$$

Using this subsequence $\{n_k\}$, we consider the sequence $\{T^{-j_{n_k}}F^{n_k}\eta_{\bar{v}}\}_{k\in\mathbb{N}}$ from which a convergent subsequence can be extracted by Helly's Selection Theorem:

$$\eta_\infty = \lim_{k\to\infty} T^{-j_{n_k}}F^{n_k}\eta_{\bar{v}}.$$

Consider now the sequence$\{T^{-m\bar{v}}F^m\eta_\infty\}_{k\in\mathbb{N}}$. It satisfies $\eta_{\bar{v}} \leqslant T^{-(m+1)\bar{v}}F^{m+1}\eta_\infty \leqslant T^{-m\bar{v}}F^m\eta_\infty$.
Hence, the following limit exists

$$\phi = \lim_{m\to\infty} T^{-m\bar{v}}F^m\eta_\infty$$

and satisfies $T^v\phi = F\phi$ and $\lim_{x\to+\infty}\phi(x) = 1$. As for the limit $\lim_{x\to-\infty}\phi(x)$, in general we cannot say more than $\lim_{x\to-\infty}\phi(x) \in \{0, c\}$.

However, if f and h are such that

$$\frac{df}{dc}(c) = +\infty \qquad \text{and} \qquad \inf\{x \in \mathbb{R} \ : \ h(x) > 0\} = -\infty,$$

then one can prove that $\lim_{x\to-\infty}\phi(x) = 0$.

Finally, to conclude in the general case, we show that every pair of bistable map f and distribution function h can be approximated pairs satisfying the previous condition. The existence of fronts then follows from continuity properties (see the references for more details).