Tangle Dynamics in a Toy Model of a Vortex Bubble

J.D. Mireles James

Rutgers University

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Outline

Talk Outline

- Introduce a simple family of quadratic volume-preserving discrete dynamical system.
- Introduce the notion of a "vortex-bubble".
- Study homoclinic tangle dynamics in the bubble of the quadratic family.

- In order to accurately and efficiently compute the stable and unstable manifolds of fixed points in the quadratic family, we use the Parameterization Method.

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The Map

The Volume Preserving Hénon Map, is given by

$$f(x,y,z) = \begin{pmatrix} z + Q_{\alpha,\tau,a,b,c}(x,y) \\ x \\ y \end{pmatrix},$$

where Q is the quadratic function

$$Q_{\alpha,\tau,a,b,c}(x,y) = \alpha + \tau x + a\,x^2 + b\,xy + c\,y^2,$$

and $a + b + c = 1$.

Ramarks

- The Volume-Preserving Hénon map (VPHM) is a normal form for the family of quadratic diffeomorphisms with quadratic inverse.
- The VPHM generalizes the classical area preserving Hénon family given by:

$$f(x, y) = \begin{pmatrix} y + (1 - ax^2) \\ x \end{pmatrix}.$$

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- If $\tau^2 > 4\alpha$ there are exactly two fixed points $p_{0,1}$ located at

$$p_{0,1} = \begin{pmatrix} x_{0,1} \\ x_{0,1} \\ x_{0,1} \end{pmatrix},$$

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$$x_{0,1} = \frac{-\tau}{2} \pm \frac{\sqrt{\tau^2 - 4\alpha}}{2}.$$

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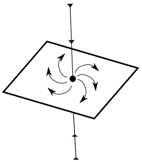

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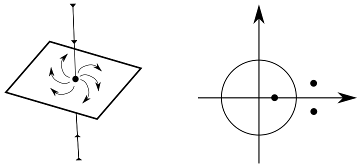

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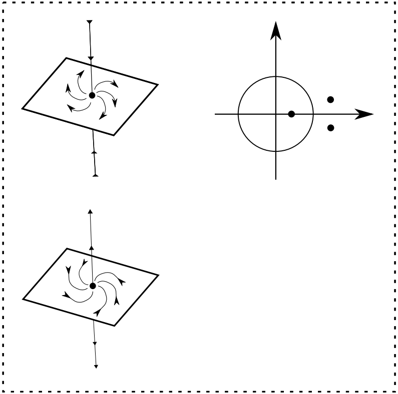

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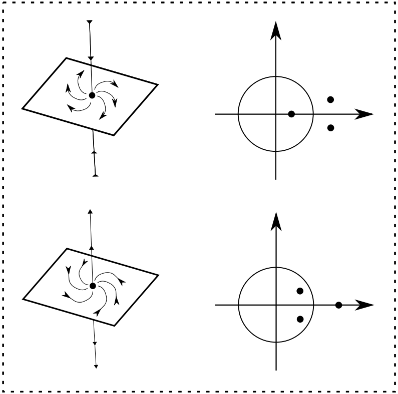

Fixed Points

- We refer to the first case as stability type $(2, 1)$, and the second case as stability type $(1, 2)$.
- On a large set of parameters, the VPHM has one fixed point p_0 of stability type $(2, 1)$ and a second fixed point p_1 of stability type $(1, 2)$.
- Then $W^u(p_0)$ and $W^s(p_1)$ are two dimensional manifolds which may intersect in such a way as to enclose a volume in phase space (Resonance Zone).
- In [DM09] it is shown that, on a parameter set of large measure, an elliptic invariant circle exists in the resonance zone. We refer to this situation a “vortex bubble”.

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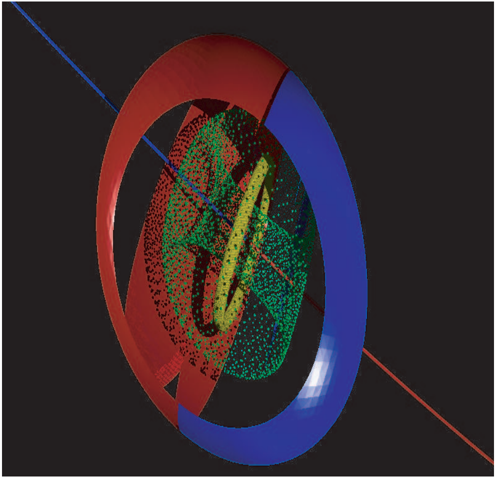
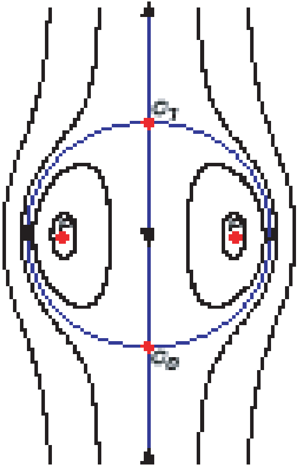
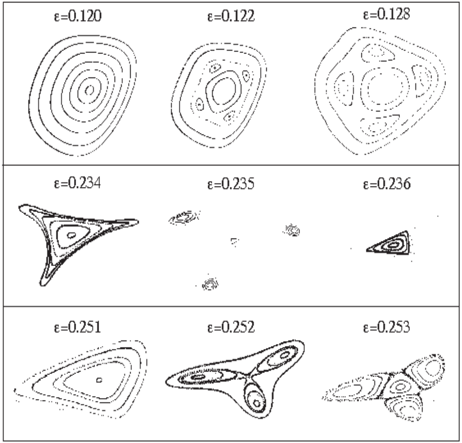
Regular/Integrable Vortex Bubble





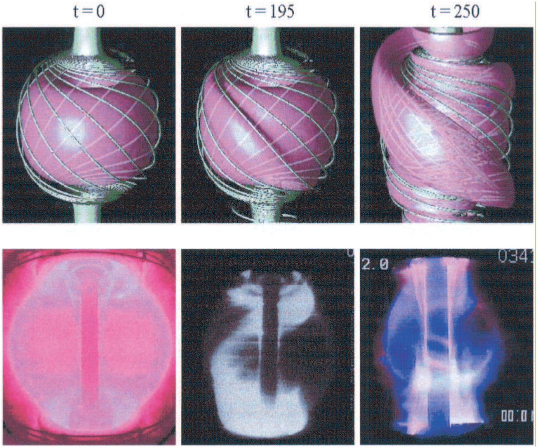
Irregular Vortex Bubble [DM09]



Some Bubbles in Physics: Hill's Vortex (flow)

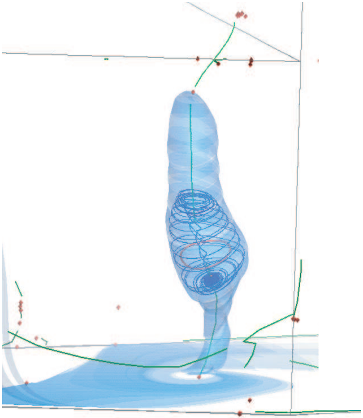
Pekert and Sadlo (2007)

Some Bubbles in Physics: Tokamak (PDE+Data)



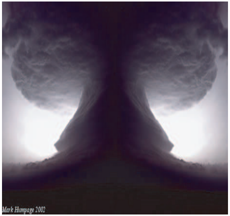
Hayashi, Nizuguchi, Sato (2001)

Some Bubbles in Physics: Draft Tube (flow)



Pekert, Sado (Peprint)

Some Bubbles in Physics: Natural Vortices



Mark Humpage and Antoniou, Lambropoulou

Remarks

- The VPHM has been proposed as a toy model for ‘vortex bubble’ dynamics.
- [DM09] presents a qualitative numerical study of elliptic dynamics inside the vortex-bubble of the VPHM (invariant circles, invariant tori, and their bifurcations).
- We are interested in studying the boundary of the bubble itself, as well as homoclinic tangle dynamics near/in the bubble.

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Stable Manifolds of fixed Points

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real analytic mapping with real analytic inverse.
- Suppose that $f(p) = p$, and that $Df(p)$ is invertible and diagnalizable.
- Suppose that $\lambda_1^s, \ldots, \lambda_k^s$ are k distinct stable eigenvalues of $Df(p)$, ($|\lambda_i^s| < 1$). Suppose that all the other eigenvalues of $Df(p)$ have norm greater than or equal to one.
- Let Λ be the $k \times k$ matrix with diagonal entries λ_i^s.
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Parameterization for Stable Manifolds

- The parameterization method consists of looking for an analytic mapping $P : B_r(0) \subset \mathbb{R}^k \to \mathbb{R}^n$ satisfying;

$$P(0) = p \qquad DP(0) = A$$

as well as

$$f[P(\theta)] = P[\Lambda\theta] \qquad \theta \in B_r(0) \tag{1}$$

We call Equation (1) the *invariance equation* for P. Note that A is not unique.

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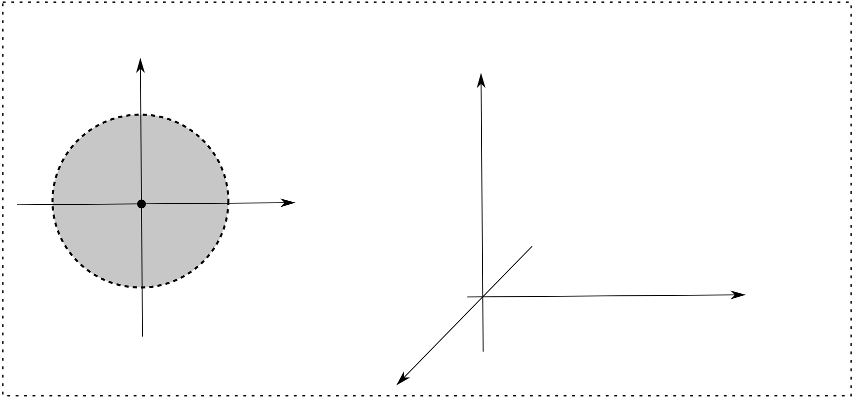

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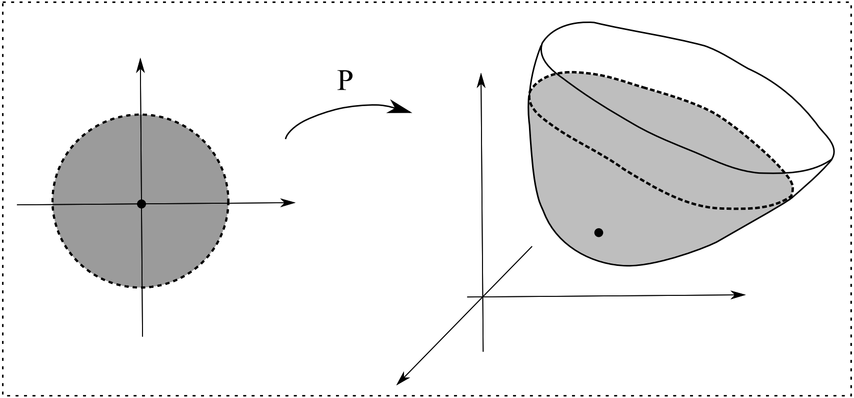

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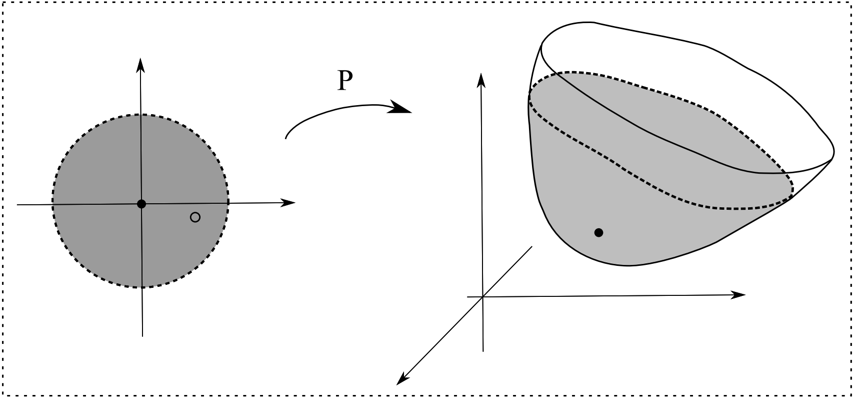

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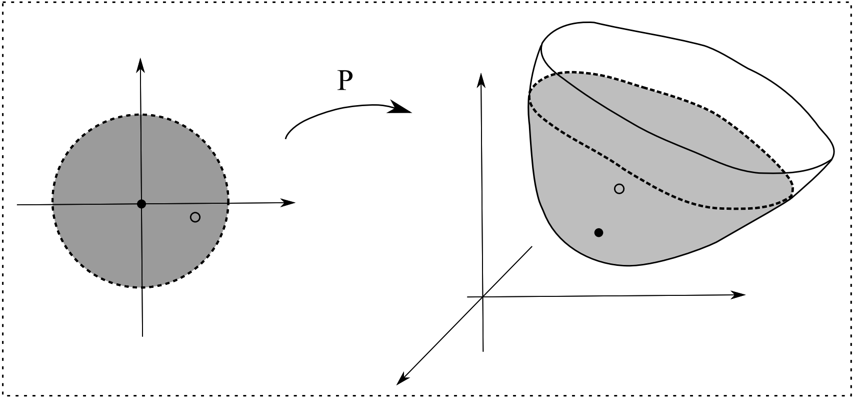

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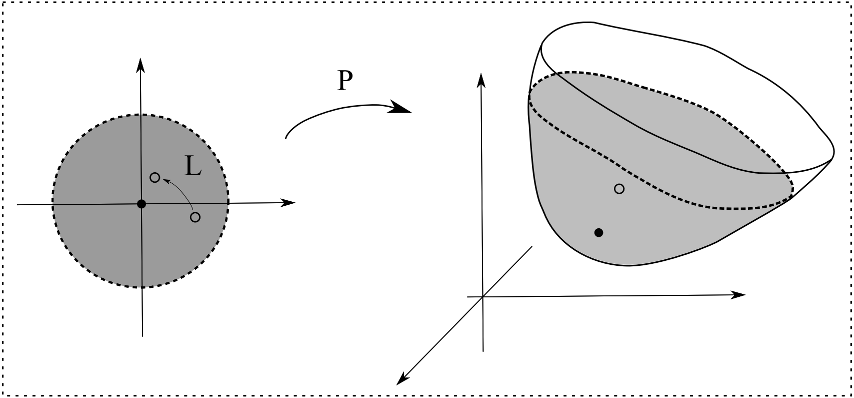

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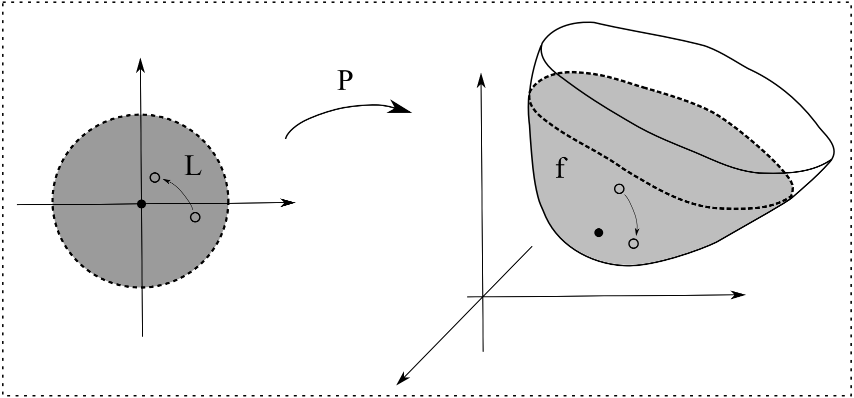

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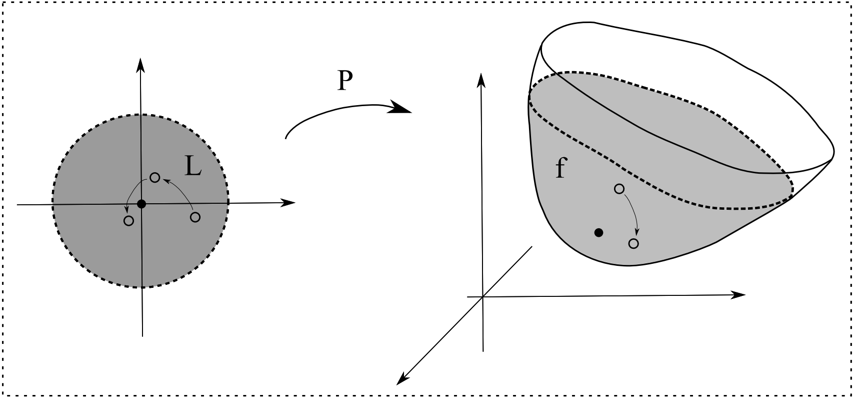

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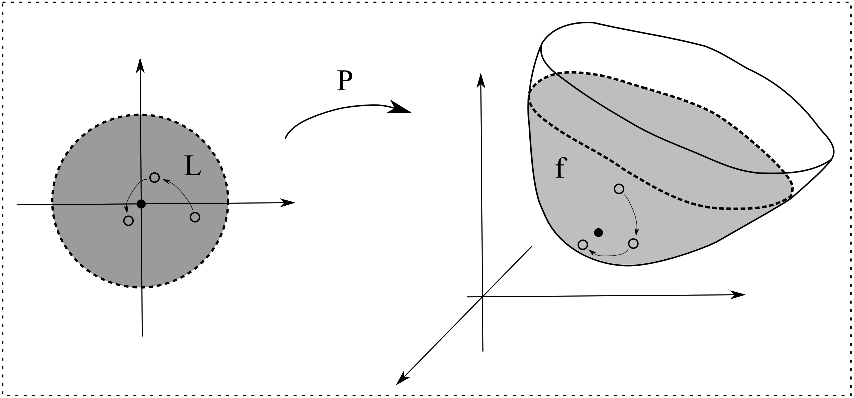

Coefficient Computation

- Since P solves a (functional) initial value problem with analytic data, assume that

$$P(\theta) = \sum_{|\alpha| \geq 0} a_\alpha \, \theta^\alpha, \qquad a \in \mathbb{R}^n,$$

  multi-index notation; so $\theta \in \mathbb{R}^k$, $\alpha \in \mathbb{N}^k$, $|\alpha| = \alpha_1 + \ldots + \alpha_k$, $\theta^\alpha = \theta_1^{\alpha_1} \cdot \ldots \cdot \theta_k^{\alpha_k}$, $a_\alpha \in \mathbb{R}^n$.
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Coefficient Computation

Example the one-dimensional stable or unstable manifolds of p_0 in the VPHM has:

$$\begin{pmatrix} \tau + 2aa_0 + bb_0 - \lambda^n & ba_0 + 2cb_0 & 1 \\ 1 & -\lambda^n & 0 \\ 0 & 1 & -\lambda^n \end{pmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} s_n \\ 0 \\ 0 \end{bmatrix},$$

where

$$s_n = -\sum_{j=1}^{n-1} \left[aa_j a_{n-j} + ba_j b_{n-j} + cb_j b_{n-j} \right].$$

Coefficient Computation

One can check that the matrix equation has the form

$$[Df(p) - \lambda^n I]\, v_n = y_n.$$

where y_n is a known quantity depending recursively on terms of order less than n.

The matrix equation can be solved uniquely as long as $\lambda^n \neq \lambda$. This condition holds for all $n \geq 2$ as $|\lambda| < 1$, so we can solve the system recursively for all the coefficients of order two and higher.

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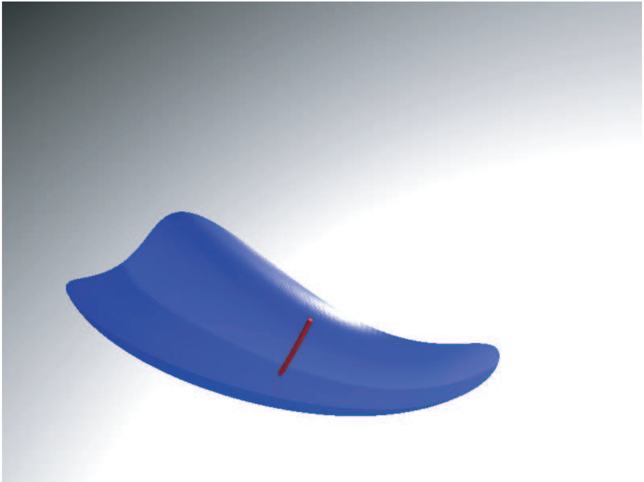
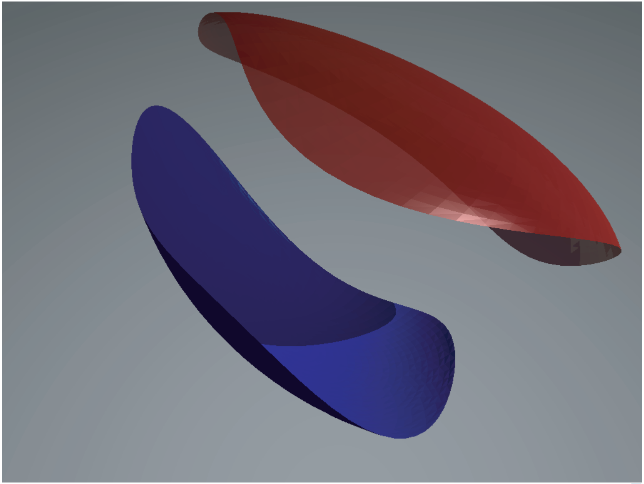
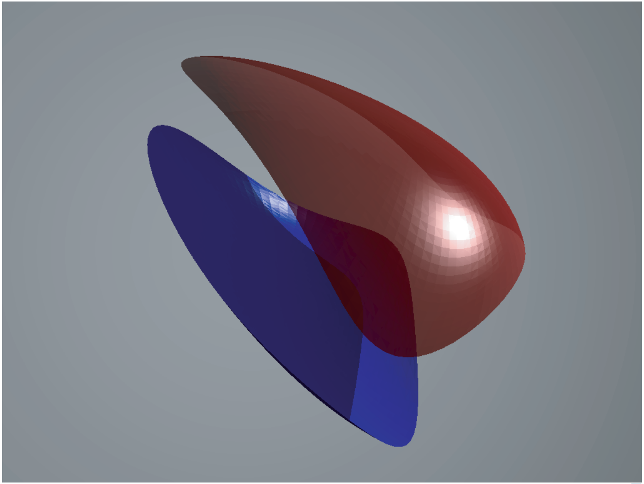
Example: $W^u_{loc}(p_0)$ and $W^s_{loc}(p_0)$



Example: $W^u_{loc}(p_0)$ and $W^s_{loc}(p_1)$



Example 1: $W^u_{loc}(p_0)$ and $W^s_{loc}(p_1)$



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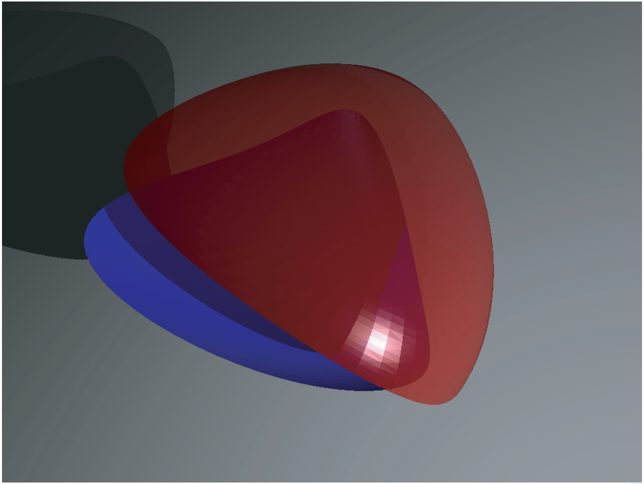
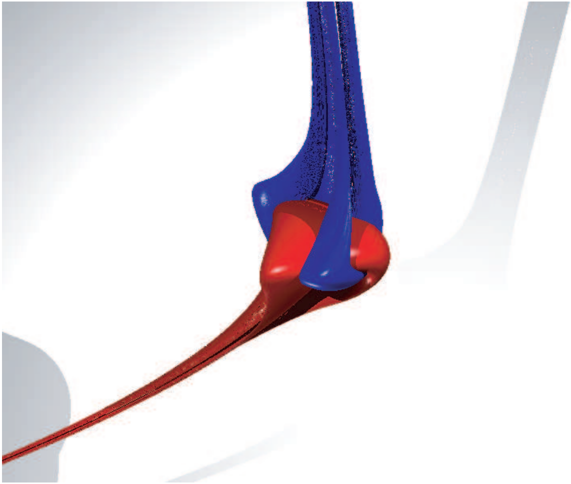


Globalization of the Stable and Unstable Manifolds :



Remarks

- The unstable manifold is similar (it is the stable manifold of f^{-1}).
- Convergence of the series is treated in the references.
- A version for flows is also in the references.

The equations determining the coefficients of the two dimensional manifolds for the VPHM, as well as numerical implementation of the parameterization computations, are in:

- **Complex Conjugate Pair** "Computation of Heteroclinic Arcs with Application to the Volume Preserving Hénon family" Hector Lomelí and JDMJ (submitted) mparc 10 10-3
- **Real Distinct** "Quadratic Volume Preserving Maps: Homoclinic Tangles" JDMD (in preparation)

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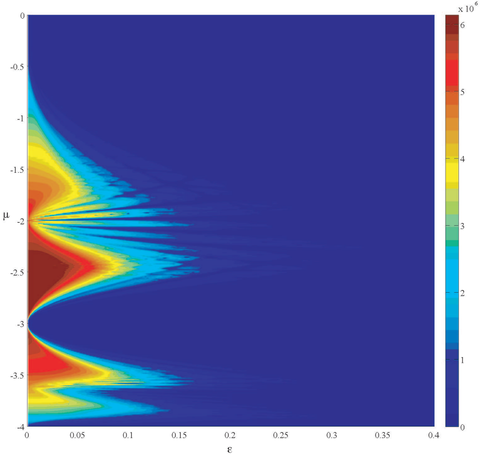
Outline

Set of Bounded Orbits [DM09]



Vortex Bubbles Bifurcations: (vertical)

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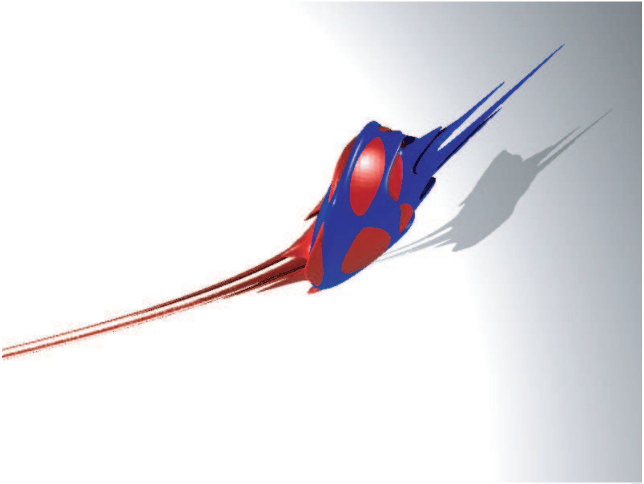

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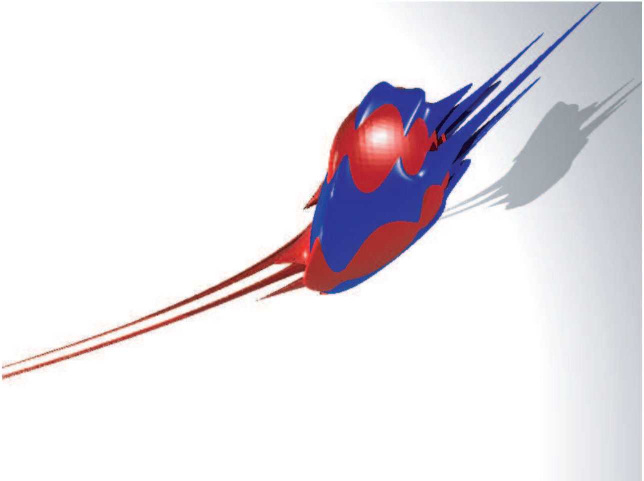

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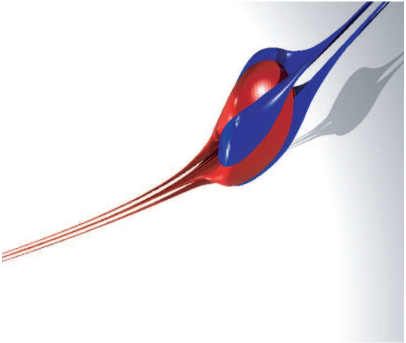

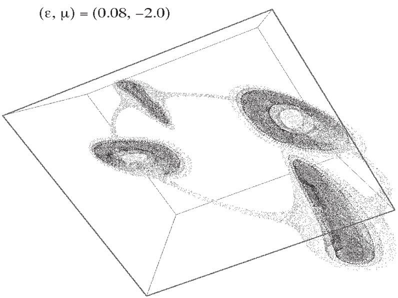

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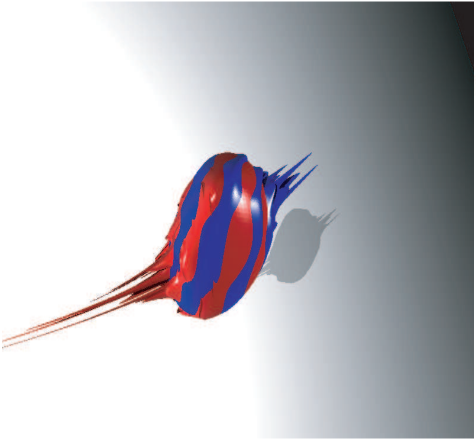

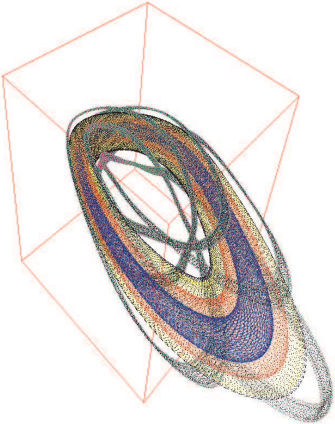

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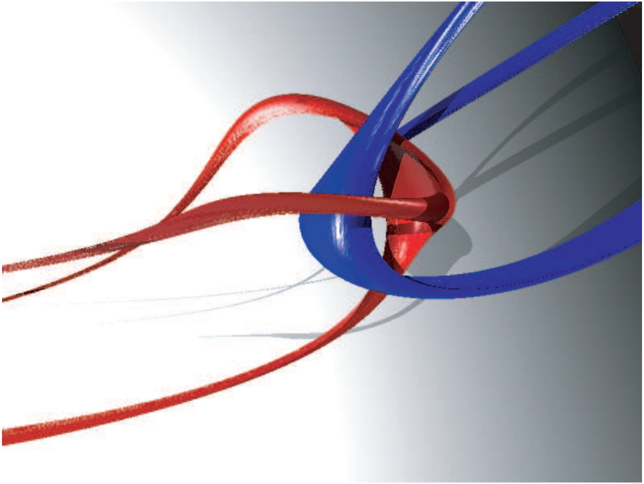

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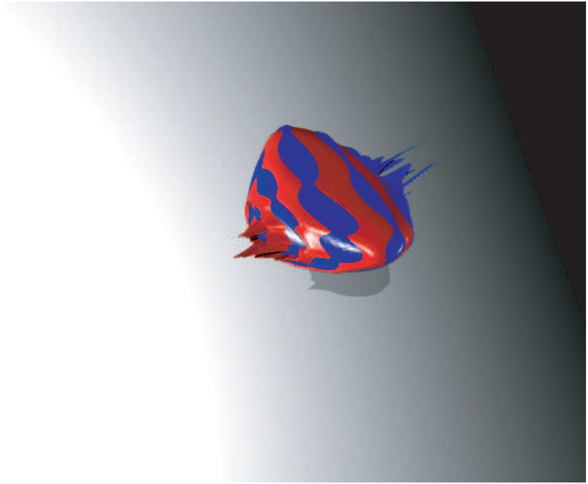

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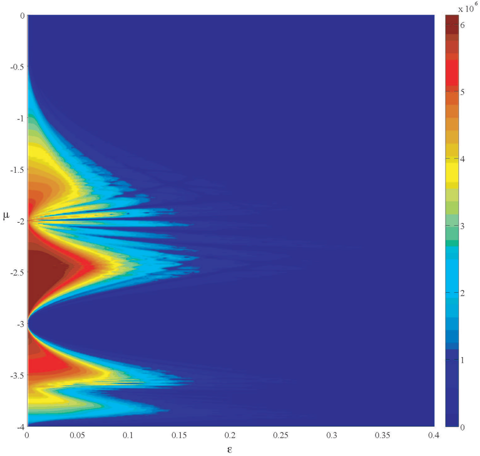
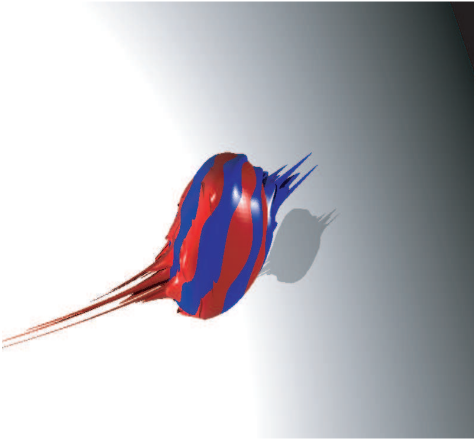
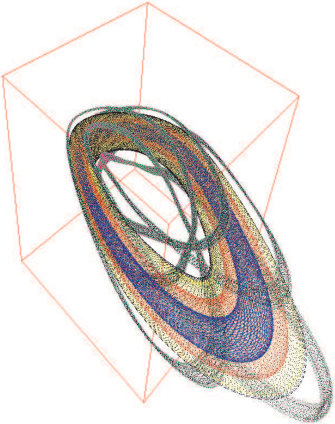


Vortex Bubbles Bifurcations: (horizontal)





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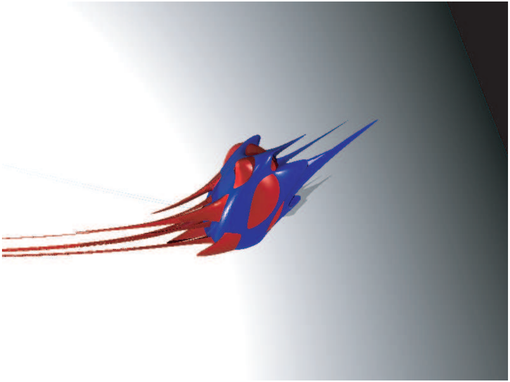

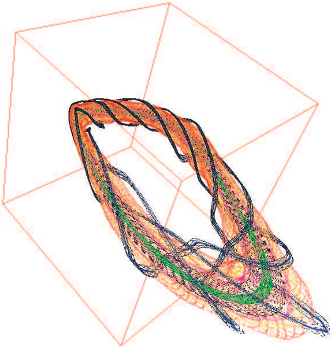

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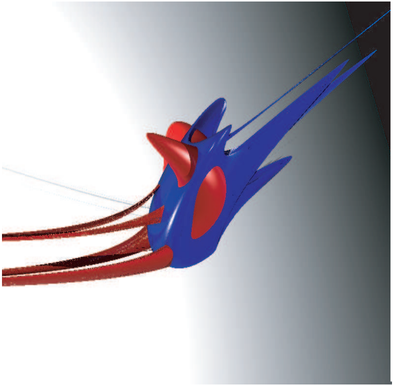

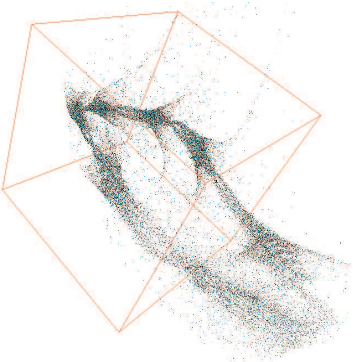

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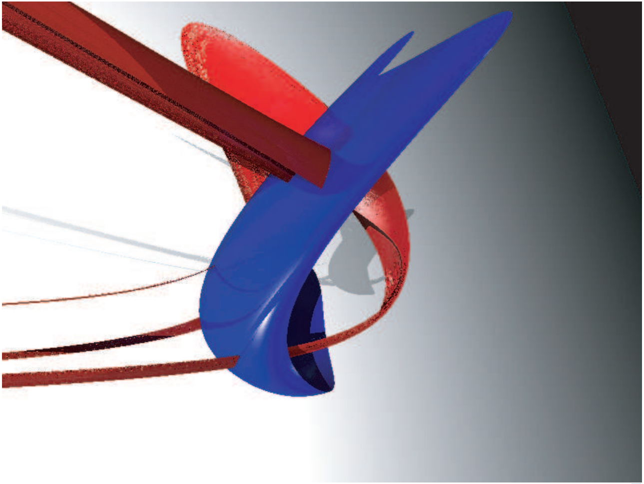
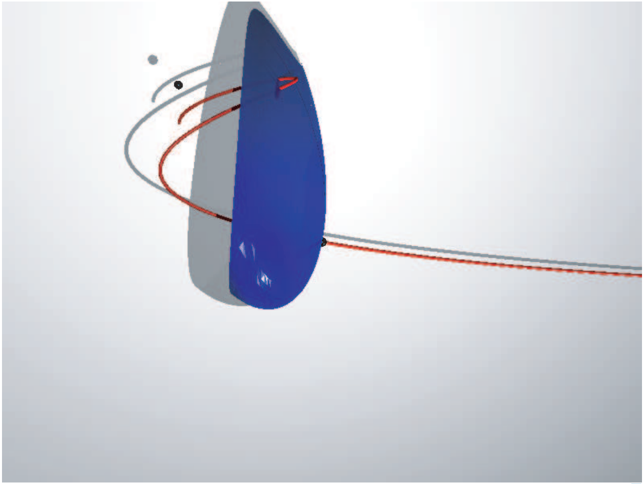
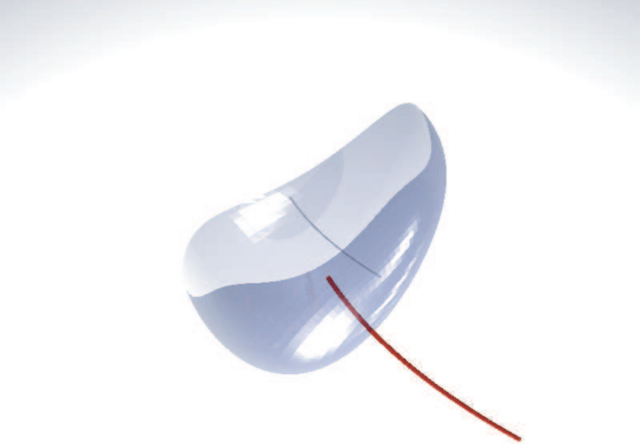


Outline

Large ϵ



Bubble Tangle



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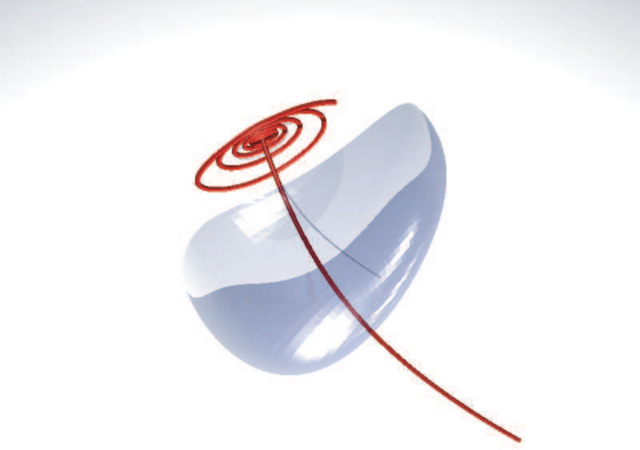
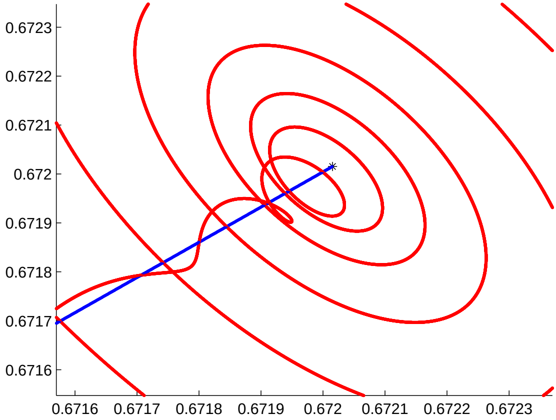
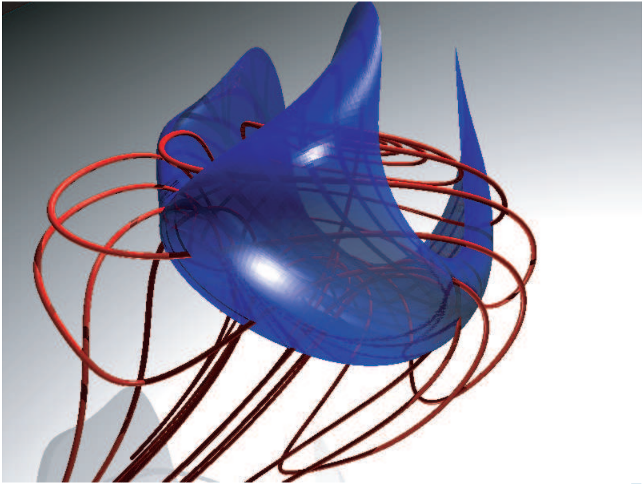


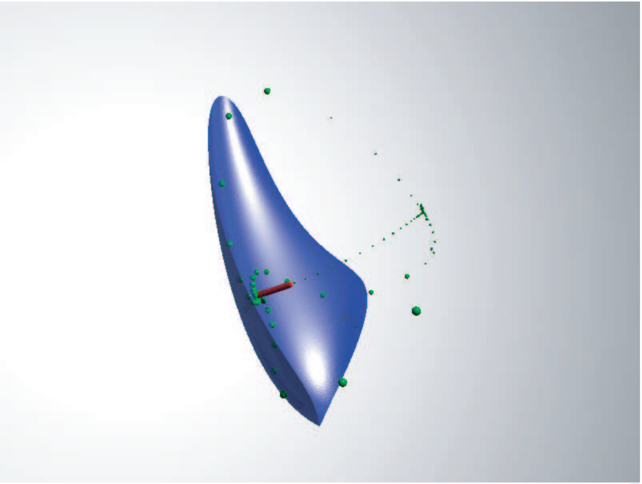
Bubble Tangle: Near p_1



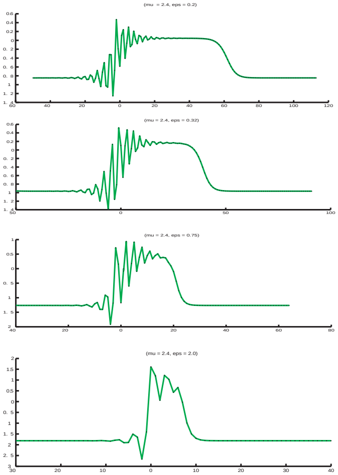
Bubble Tangle: Global



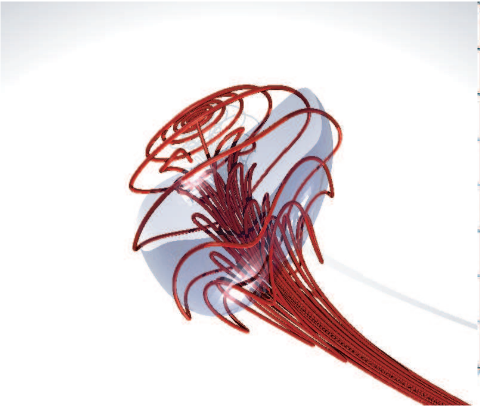
Bubble Tangle: homoclinic orbit

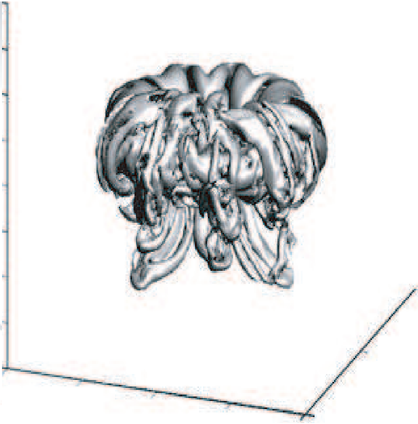


Bubble Tangle: vary ϵ

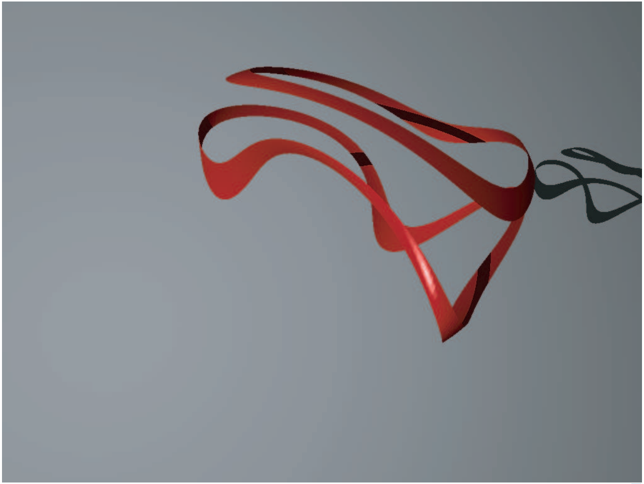


Bubble Tangle: "Hairpin"





3D Tangle Geometry



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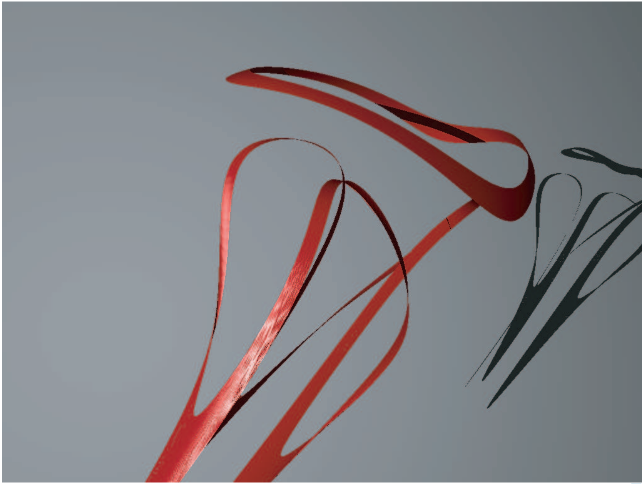

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