Shadowing orbits for dissipative PDEs

(with G. Arioli)

(R) **A few references**

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The methods apply in principle to general **dissipative evolution equations**

$$\dot{\boldsymbol{u}} + (-\boldsymbol{\Delta})^m \boldsymbol{u} + \boldsymbol{H}_\alpha(\boldsymbol{u}, \nabla \boldsymbol{u}, \ldots) = \boldsymbol{0}\,,$$

for sufficiently simple domains and analytic nonlinearities H.

Start with **stationary solutions** $\dot{u} = 0$ and rewrite the resulting equation as

$$F_\alpha(u) = u\,, \quad \text{where} \quad F_\alpha(u) = -(-\Delta)^{-m} H_\alpha(u, \nabla u, \ldots)\,,$$

The idea is to exploit the compactness of $(-\Delta)^{-m}$ to obtain good finite dim approximations.

Example. The one-dimensional **Kuramoto-Sivashinsky** (KS) equation

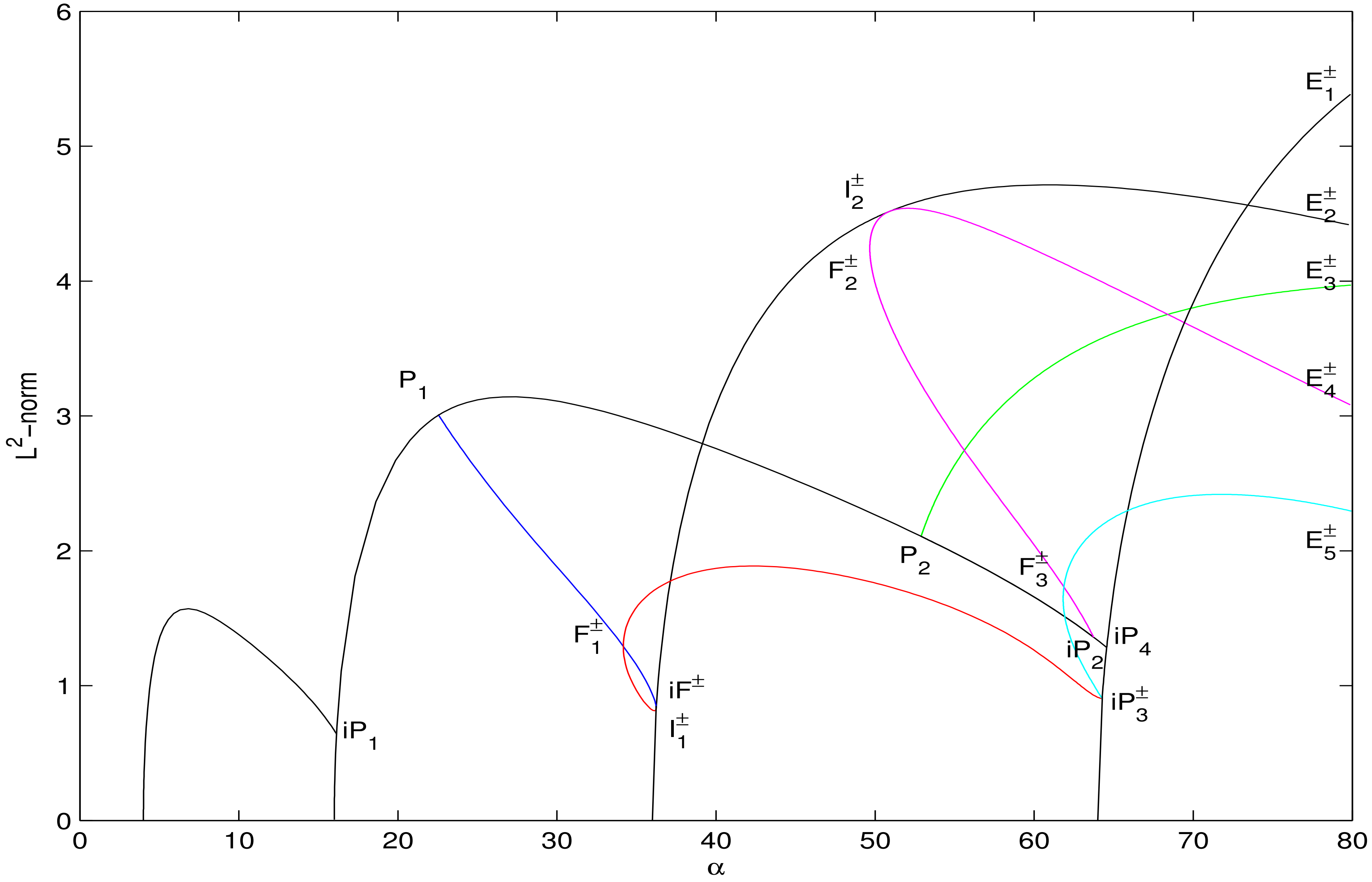
$$\partial_t u + 4\partial_x^4 u + \alpha\big(\partial_x^2 u + 2u\partial_x u\big) = 0\,, \qquad t \geq 0, \quad x \in [0, \pi]\,,$$

with homogeneous Dirichlet boundary conditions.

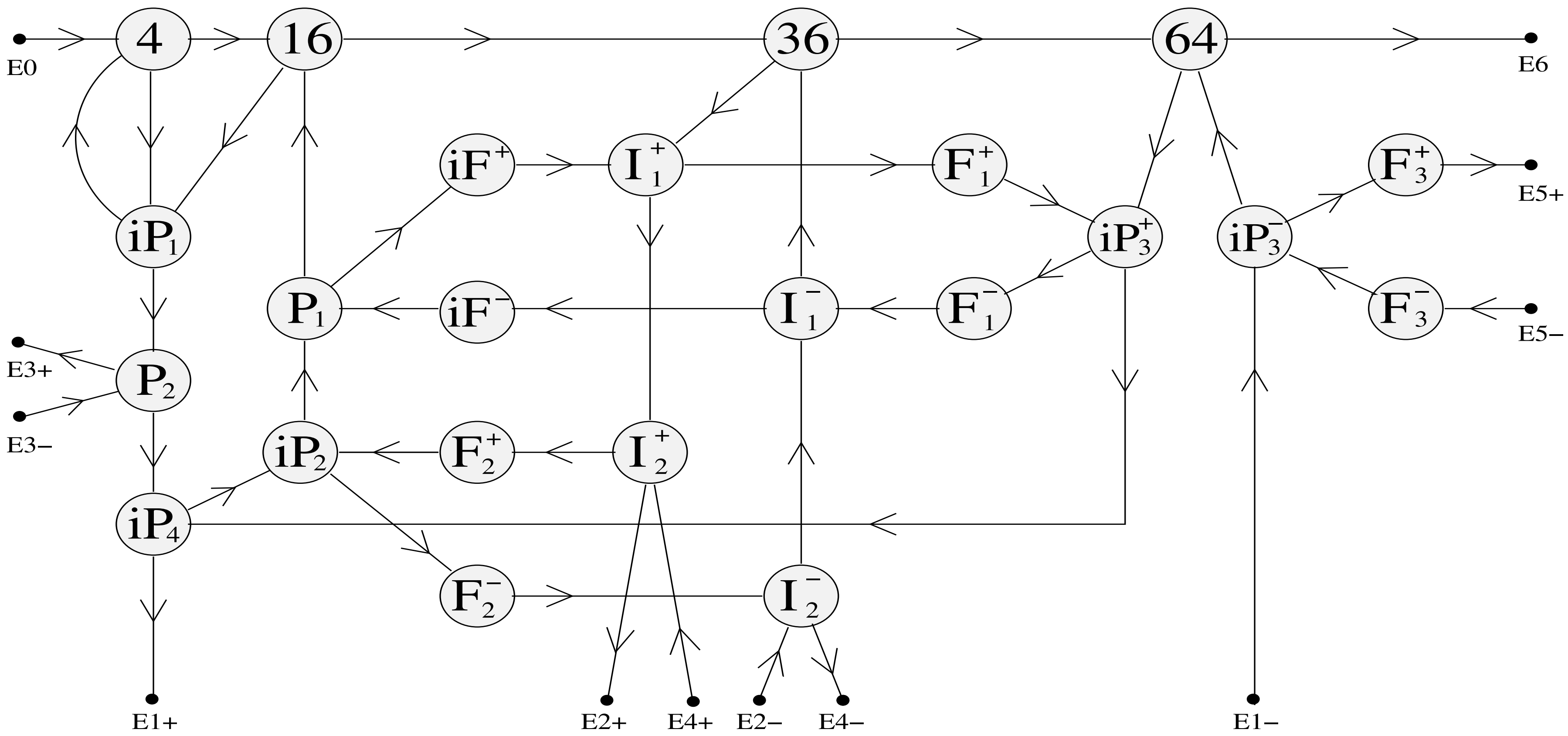
A trivial solution is $\boldsymbol{u} = \boldsymbol{0}$, for any value of α. It defines a line in the space of pairs (α, u) satisfying $F_\alpha(u) = u$. Other solutions **bifurcate** off this line at $\alpha = 4k^2$, with k a positive integer. The resulting solution curves bifurcate again ...

Determining the **bifurcation diagram** is simplified by the fact that many bifurcations involve the breaking of some symmetry.

Bifurcation diagram (L^2 norm versus α) for the **Kuramoto-Sivashinsky** equation



Theorem. [G. Arioli, H.K.] *For $0 \le \alpha \le 80$, the stationary KS equation equation exhibits eleven pitchfork bifurcations (4, 16, 36, 64, P_1, P_2, iP_1, iP_2, $iP_3^{\pm}$, iP_4), four intersection bifurcations ($I_1^{\pm}$, $I_2^{\pm}$), and eight fold bifurcations ($F^{\pm}$, $F_1^{\pm}$, $F_2^{\pm}$, $F_3^{\pm}$), connected by 44 smooth solution curves, as depicted below. These curves undergo no other bifurcations for $0 \le \alpha \le 80$.*



Other results include bounds on the values of α for each bifurcation point, as well as the dimension of the unstable manifold and the L^2 norm of 30 selected solutions.

Non-stationary orbits. The goal is to solve initial value problems of the form

$$\dot{u} = Lu + G(u)\,, \qquad u(0) = \nu\,,$$

where L is linear and "very negative". Rewriting the equations as

$$\partial_t \left[e^{-tL}u\right] = e^{-tL}G(u)\,, \qquad u(0) = \nu\,,$$

integrating both sides, and then multiplying by e^{tL}, we get the integral equation

$$u(t) = e^{tL}\nu + \int_0^t e^{(t-s)L}G(u(s))\,ds\,. \tag{$\circledast$}$$

The flow Φ. After defining a suitable Banach space $\mathcal{X}$ of admissible initial conditions ν, solve the equation by iteration, on a space of continuous curves $u : [0,T] \to \mathcal{X}$. This yields the time-$t$ maps $\Phi_t(\nu) = u(t)$ for times up to T.

Problem: Computer-estimates only work for small $T > 0$.
Even composition of time-t maps gets soon out of control.

Way out: Shadowing of an approximate numerical orbit, using a sequence of boxes . . .

Non-stationary solutions for KS. The most interesting are probably chaotic orbits, such as the ones found numerically in [F. Christiansen, P. Cvitanovic, V. Putkaradze, '97] for $\alpha \approx 137$.

For "simplicity", we focus on **periodic orbits**.
Such orbits have also been constructed in [P. Zgliczyński , preprint '08], using different methods.

Recall KS:

$$\partial_t u = \mathcal{L}u - \alpha\partial_x(u^2)\,, \qquad \mathcal{L} = -4\partial_x^4 - \alpha\partial_x^2\,.$$

Our "standard form" is obtained by splitting $\mathcal{L} = L + L'$, with $L < 0$, and rewriting

$$\boldsymbol{\partial_t u = Lu + G(u)\,, \qquad G(u) = L'u - \alpha\partial_x(u^2)\,.}$$

Theorem. [G. Arioli, H.K.] *The KS equation for $\alpha = 150$ has a hyperbolic periodic orbit with period $\tau = 0.00214688\ldots$. Some associated Poincaré map has a simple eigenvalue μ_1 of modulus $|\mu_1| > 4.8$, and the remaining eigenvalues $\mu_2, \mu_3, \ldots$ lie in the disk $|\mu_n| < 0.69$.*

Remarks.

- The derivative of the flow is estimated via the corresponding integral operator.
- The spaces used are far from optimal.
- The shadowing procedures uses $M = 4293$ rectangular boxes.

Sketch of the general procedure

The flow Φ. Convert the integral equation (❋) to a fixed point equation for curves $u : [0,T] \to \mathcal{X}$. Evaluating the solution u at time t defines the time-t map, $\boldsymbol{\Phi_t(\nu) = u(t)}$.
Here, t can be replaced by an interval.

Local Poincaré map. Given a codimension one affine subspace S transversal to the flow, define $\boldsymbol{P(\nu) = \Phi_{t(\nu)}(\nu)}$, with $t(\nu)$ the smallest time $t > 0$ where $\Phi_t(\nu) \in S$.
A bound on $t(\nu)$ is an interval $[a,c]$ such that $\Phi_a(\nu)$ and $\Phi_c(\nu)$ lie on different sides of S.
Then $\Phi_{[a,c]}(\nu)$ is an enclosure for $P(\nu)$.

The local Poincaré maps P_j. Using an approximate orbit $t \mapsto \bar{u}(t)$, choose M milestones $\bar{u}_j$ along this orbit, and Poincaré section $\boldsymbol{S_j = \bar{u}_j + X_j}$ transversal to $\bar{u}$. Define $\boldsymbol{P_j : S_{j-1} \to S_j}$ as above.

Shadowing. In each section S_j choose an appropriate box B_j and check covering condition for $P_j(B_{j-1})$ and B_j . Here we use the **derivative of** $\boldsymbol{\Phi_t}$.
In the periodic case ($j = 0$ is identified with $j = M$) this implies the existence of a fixed point for the full Poincaré map $\boldsymbol{\Psi = P_M \circ \ldots \circ P_2 \circ P_1}$ and a closed orbit u for the flow.

Linearized Poincaré maps. Let u_j be the point where the orbit u intersects S_j. Estimating the velocities $\dot{u}_j = Lu_j + G(u_j)$ gives bounds on the **derivatives** $\boldsymbol{DP_j(u_{j-1})}$.

Hyperbolicity. Check cone conditions (linear analogue of covering conditions) for each $DP_j(u_{j-1})$. Then $D\Psi(u_0)$ satisfies a cone condition, and hyperbolicity follows.

General framework

Integration. Rewrite (❄) as fixed point problem for $K_\nu(w) = w$ for $w(t) = u(t) - e^{tL}\nu$, where

$$\big(\boldsymbol{K_\nu(w)}\big)(\boldsymbol{t}) = \int_0^t \boldsymbol{e^{(t-s)L} G\big(w(s) + e^{sL}\nu\big)\, ds}\,, \qquad \boldsymbol{0 \le t \le T.}$$

Since the integrand can vary rapidly in t near $t = 0$, partitioning $J = [0, T]$ into n subintervals $J_i = [t_{i-1}, t_i]$, with the partition being **finer near $\boldsymbol{t_0 = 0}$**, than near $t_n = T$, and . . .

Assuming the eigenfunction $\{\boldsymbol{v}_k\}$ of L span a dense subspace of $\mathcal{X}$, define $\mathcal{C}(J, \mathcal{X})$ to be space of all functions $w(t) = \sum_k w_k(t)\boldsymbol{v}_k$ that have continuous coefficients $w_k : J \to \mathbb{R}$, and a finite norm

$$\|w\| = \max_i \sum_k \sup_{t \in J_i} \|w_k(t)\boldsymbol{v}_k\|\,.$$

The following is **specific to KS** (with $\mathcal{X}$ defined later).

Lemma 1. *K_ν is a compact map on $\mathcal{C}(J, \mathcal{X})$, has a unique fixed point w for each $\nu \in \mathcal{X}$, and the map $\nu \mapsto w$ is of class C^1. The flow $(t, \nu) \mapsto u(t)$ is of class C^1 and compact, for $0 < t \le T$.*

Shadowing. In the case of a single expanding direction, we can use the following

Lemma 2. *Consider a Banach space* $X = \mathbb{R} \oplus Z$*, and let* V *be the closed unit ball in* Z*. Let* F *be a continuous and compact map*

$$[-1,1] \times V \xrightarrow{F} \mathbb{R} \times V\,,$$

$$[-1,-\vartheta] \times V \xrightarrow{F} (-\infty,-1] \times V\,,$$

$$[\vartheta,1] \times V \xrightarrow{F} [1,\infty) \times V\,,$$

for some positive $\vartheta \le 1$*. Then* F *has a fixed point in* $[-\vartheta,\vartheta] \times V$*.*

Assume $\mathcal{X} = \mathcal{Y} \oplus \mathcal{V} \oplus \mathcal{Z}$, where $\mathcal{Y}$ and $\mathcal{V}$ are one-dimensional subspaces of $\mathcal{X}$
(in our case roughly the unstable and velocity directions of the flow).
Denote by U and V the closed unit balls in $\mathcal{Y}$ and $\mathcal{Z}$, respectively.

Definition. *A* section *(of* $\mathcal{X}$*) is codimension one affine subspace of* $\mathcal{X}$*. A* box *in a section* S *is the image of* $U \times V$ *under a bi-continuous affine map* $\psi : \mathcal{Y} \oplus \mathcal{Z} \to S$*.*

Definition. *Let* $B_i = \psi_i(U \times V)$ *and* $B_j = \psi_j(U \times V)$ *be boxes in two section* S_i *and* S_j*, respectively. Given a map* $f : B_i \to S_j$*, we say that* B_i f*-covers* B_j *if the map* $F : U \times V \to \mathcal{Y} \oplus \mathcal{Z}$*, defined by* $F = \psi_j^{-1} \circ f \circ \psi_i$*, satisfies the hypotheses of of Lemma 2, for some* $\vartheta < 1$*.*
For simplicity, we identified here $\mathcal{Y}$ *with* $\mathbb{R}$*, and* U *with* $[-1,1]$*.*

Corollary 3. *If for each* j*, the box* B_{j-1} P_j*-covers* B_j*, then the Poincaré map* $\Psi : S_0 \to S_0$*, defined by* $\Psi = P_M \circ \ldots \circ P_2 \circ P_1$*, has a fixed point in* B_0*.*

Linearized Poincaré map at $u_{j-1} \in S_{j-1}$.

$$\boldsymbol{DP_j(u_{j-1})w = D\Phi_{t(u_{j-1})}(u_{j-1})w - \frac{\eta_j(D\Phi_{t(u_j)}(u_{j-1})w)}{\eta_j(\dot{u}_j)}\,\dot{u}_j\,.}$$

Here, $\dot{u}_j = Lu_j + G(u_j)$ is the velocity at $u_j = P_j(u_{j-1})$.
And η_j is the linear functional that defines the hyperplane X_j at the section $S_j = \bar{u}_j + X_j$.

Consider now the points u_j where the periodic orbit intersects the Poincaré planes S_j .
The *low-frequency parts* $\boldsymbol{\ell_j = \mathbb{P}_L \dot{u}_j}$ are estimated explicitly in our construction of the orbit.
To estimate the *high-frequency parts* $\boldsymbol{h_j = \mathbb{P}_H \dot{u}_j}$ use that $\dot{u}_j = D\Phi_{t(u_{j-1})}(u_{j-1})\dot{u}_{j-1}$.

Lemma 4. *Let* $k_j = \mathbb{P}_H D\Phi_{t(u_{j-1})}(u_{j-1})\ell_{j-1}$ *and* $D_j = \mathbb{P}_H D\Phi_{t(u_{j-1})}(u_{j-1})\mathbb{P}_H$. *Then*

$$\|h_j\| \le \|k_j\| + \|D_j\|\|h_{j-1}\|\,, \qquad j = 1, 2, \ldots, M.$$

In particular, if $\|k_j\| \le b$ *and* $\|D_j\| \le a < 1$ *for all* j, *then* $\|h_j\| \le (1-a)^{-1}b$.

Hyperbolicity. In the case of a single expanding direction, we can use the following

Lemma 5. *Let $A \neq 0$ be a bounded linear operator on a real Banach space $X = Y \oplus Z$, with Y one-dimensional. Thus, if $y \in Y$ and $z \in Z$, we have a unique decomposition*

$$A(y+z) = y' + z' \,, \qquad y' \in Y,\ z' \in Z \,.$$

Assume now that A is compact, and that there exists positive real numbers $\beta < \alpha$, such that $\|z'\| \leq \beta \max\{\|y\|, \|z\|\}$, and such that $\|y'\| \geq \alpha\|y\|$ whenever $\|y\| \geq \|z\|$. Then A has a simple eigenvalue λ of modulus $|\lambda| \geq \alpha$, and no other eigenvalue of modulus $> \beta$.

Definition. *Let $X = \mathcal{Y} \oplus \mathcal{Z}$, and let $\alpha > \beta$ be positive real numbers. Given two sections $\bar{u}_i + \psi_i(X_i)$ and $\bar{u}_j + X_j = \psi_j(X)$ of $\mathcal{X}$, and a linear map $B : X_i \to X_j$, we say that B satisfies the (α, β) cone condition , if $A = D\psi_j^{-1} B D\psi_i$ satisfies the hypotheses of Lemma 5.*

Consider again the local Poincaré maps $P_j : S_{j-1} \to S_j$ described earlier.
Denote by u_j the intersection of the periodic orbit with the Poincaré plane S_j.

Corollary 6. *If for each j, the derivative $DP_j(u_j)$ satisfies a (α_j, β_j) cone condition, then $D\Psi(u_0)$ has a simple eigenvalue μ_1 of modulus $|\mu_1| \geq \prod_j \alpha_j$ and no other spectrum outside the disk $|\mu| \leq \prod_j \beta_j$.*

The KS equation $\boldsymbol{\partial_t u = Lu + G(u)}$. Recall that

$$\boldsymbol{G(u) = L'u - \alpha\partial_x(u^2)\,, \qquad L + L' = -4\partial_x^4 - \alpha\partial_x^2\,,}$$

with Dirichlet boundary conditions on $[0, \pi]$. The eigenvalues of $-L$ and $-L'$ are

$$\lambda_k = \begin{cases} 0, & \text{if } k \le \kappa; \\ 4k^4 - \alpha k^2 & \text{if } k > \kappa; \end{cases} \qquad \lambda'_k = \begin{cases} 4k^4 - \alpha k^2 & \text{if } k \le \kappa; \\ 0 & \text{if } k > \kappa; \end{cases} \tag{0.1}$$

with eigenvectors $\boldsymbol{v_k(x) = \sin(kx)}$. Here, $\kappa \ge \sqrt{\alpha}/2$, so that $\alpha k^2 - 4k^4 \le 0$ for $k \ge \kappa$.

Function space used: $\mathcal{X} = \mathcal{X}_1^o$ with $\rho = 2^{-7}$.

Given $\rho > 0$, and a nonnegative integer K, define

$\boldsymbol{\mathcal{X}_K^o}$: Space of odd **2π-periodic real analytic functions** on the strip $|\mathrm{Im}\, x| < \rho$,

$$u = \sum_{k \ge K} u_k \boldsymbol{v}_k\,, \qquad \|u\| \stackrel{\text{def}}{=} \sum_{k \ge K} |u_k| e^{\rho k} < \infty\,.$$

$\boldsymbol{\mathcal{X}_K^e}$: Analogous space of even 2π-periodic functions.

The computer-assisted proof uses a type

`Ball`: $\texttt{S} = (\texttt{S.C}, \texttt{S.R}) \in \texttt{Rep} \times \texttt{Radius}$.
representing intervals in $\mathbb{R}$, or balls in a Banach space X,

$$\mathcal{B}(\texttt{S}) = (\texttt{S.C}) + (\texttt{S.R})\mathbb{U}_{\mathbb{R}}\,, \qquad \mathcal{B}(\texttt{S}, X) = (\texttt{S.R})\mathbb{U}_X\,.$$

where $\mathbb{U}_X = \{x \in X : \|x\| \le 1\}$.

The **representable sets** in $\mathcal{X}_1^o$ are taken to be of the form

$$\mathcal{B}(\texttt{F}) = \sum_{K=1}^{D} \mathcal{B}(\texttt{F.C(K)})\sin(\texttt{K.}) + \sum_{K=1}^{2D} \mathcal{B}(\texttt{F.E(K)}, \mathcal{X}_K^o)\,, \qquad \texttt{F} \in \texttt{SFourier}\,.$$

The representable sets in $\mathcal{X}_0^e$ are defined analogously. Both are associated with data of type `SFourier`, which is is an instantiation (`FCoeff` ⇒ `Ball`) of

`Fourier: F=(F.T,F.C,F.E)`, with `F.T` encoding the type (even or odd, domain ρ), and

```
    F.C: array [0..D] of FCoeff;
    F.E: array [0..2*D] of FCoeff;
```

Implement **bounds** hierarchically, starting with simple and/or generic types, then for more complex types; first for basic operations, then for functions like F_α .

$\mathcal{C}(J, \mathcal{X}_K^o)$ is the Banach space of all continuous functions $w : J \to \mathcal{X}_K^o$ with ...

$$w(t) = \sum_{k \ge K} w_k(t) \boldsymbol{v}_k\,, \quad \|w\| = \max_i \|w\|_i\,, \quad \|w\|_i = \sum_{k \ge K} e^{\rho k} \max_{t \in J_i} |w_k(t)|\,.$$

Simple representable sets for these spaces associated with data of type

`ContFun`: `P=(P.C,P.E)`, where

```
P.C: array [0..PDeg] of Ball;
P.E: array [1..NErr] of Ball;
```
(nonnegative)

$\mathcal{B}$(`P.C`) : all polynomials of degree ≤ `PDeg`, whose `K`-th coefficient belongs to $\mathcal{B}$(`P.C(K)`, $\mathbb{R}$).
The polynomials on $J = [0, T]$ are expanded about $\frac{2}{3}T$.

$\mathcal{B}$(`P.E`, $\mathcal{X}_K^o$) : all functions $v \in \mathcal{C}(J, \mathcal{X}_K^o)$ such that $\|v\|_{\mathtt{I}} \le$ `P.E(I).R` for all `I`.

The representable sets for $\mathcal{C}(J, \mathcal{X}_1^o)$ are associated with data of type `TFourier`, which is an instantiation (`FCoeff` ⇒ `ContFun`) of `Fourier`. In other words,

$$\mathcal{B}(\mathtt{F}) = \sum_{K=0}^{D} \mathcal{B}(\mathtt{F.C(K)}) \sin(\mathtt{K.}) + \sum_{K=0}^{2D} \mathcal{B}(\mathtt{F.E(K)}, \mathcal{X}_K^o)\,, \qquad \mathtt{F} \in \mathtt{TFourier}\,.$$

The representable sets for $\mathcal{C}(J, \mathcal{X}_0^e)$ are defined analogously.

Now implement bounds `Contr`, `DContr`, `ContrFix`, `DContrFix`, `Phi`, `DPhi`, ...
on the maps K_ν , $\partial_\nu K_\nu$, ...

To obtain decent error bounds for `Contr`, we decompose $K_\nu(w) = P(\nu,w) + Q(\nu,w)$, where P is linear and Q quadratic,

$$Q(\nu,w) = -\alpha \int_0^t e^{(t-s)L^+} \partial_x \Big[w + e^{sL}\nu \Big]^2 .$$

Then split Q into terms $Q^{(n)}$ that are homogeneous of degree n in w. After rewriting the result in terms of Fourier coefficients, we end up with integrals like

$$\big(Q_m^{(1+)}(\nu,w)\big)(t) = -\alpha m \sum_{k+\ell=m} \nu_k \int_0^t e^{-\lambda_m(t-s)} e^{-\lambda_k s} w_\ell(s)\, ds ,$$

and use estimates like

$$\big\| Q^{(1+)}(\nu,w) \big\|_i \le 2\alpha \|\nu\| \|w\| \sup_{\substack{k\in\mathcal{K}\\ \ell\in\mathcal{L}}} \left[\frac{k+\ell}{(\lambda_{k+\ell}-\lambda_k) + 2/t_i} \right] e^{-\lambda_k t_{i-1}} .$$

Here, $\mathcal{K}$ and $\mathcal{L}$ are the frequency ranges for ν and w, respectively.
The **sup** is estimated by the program (beforehand), using monotonicity properties of [...].

`ContrFix` first computes an approximate fixed point w for K_ν . Then it encloses w in successively larger sets $\mathcal{B}(\texttt{F})$ until one of them is mapped into itself by `Contr`.

The same strategy is applied for `DContr` and `DContrFix`.

Evaluating the result of `ContrFix` at a specified time $t \in J$
yields a bound `Phi` on the flow $\Phi : (t,\nu) \mapsto u(t)$.

As much as possible of the above is kept hidden at the higher "dynamical systems" level.

The package `Boxes` uses data types `Vec` to describe sets in $\mathcal{X}_1^o$.
`V(1..N)` contains bounds on the first `N` Fourier coefficients, and
`V(N+1)` is a bound on the norm of all "higher order" terms.

Other data types include `LBasis`, `Frame`, `Box`, `TBox`, ...

Roughly speaking, a `Box` represents a set $B = C + L(R_1) \times R_2 \times H$, with
$\boldsymbol{C}$: the center of the `Box`.
$\boldsymbol{L(R_1)}$: the image of $R_1 = [-1,1] \times \{0\} \times [-1,1]^{M-2}$ under an linear transformation L on $\mathbb{R}^M$.
$\boldsymbol{R_2}$: a rectangle $R_2 = [-r_{M+1}, r_{M+1}] \times \ldots \times [-r_N, r_N]$.
$\boldsymbol{H}$: a "higher order" ball.

The zero-thickness direction of $L(R_1)$ corresponds to the Poincaré section.

A bound on the **local Poincaré map** is obtained by determining
a time interval $\mathtt{T} = [t - \varepsilon, t + \varepsilon]$ such that the flow-images of B at the two times $t \pm \varepsilon$
lie on opposite sides of the Poincaré section (at the destination point).

The 4293 boxes used in our shadowing procedure have been determined numerically.

The **box directions** fall into 4 classes.

low: The first 8 directions are roughly eigendirections of the return map (for the entire orbit).
mid-low: The next 12 directions are $(\mathbb{I} - P)\sin(k\,\boldsymbol{.})$, for $k = 9, 10, \ldots, 20 = M$.
Here, P is an approximation to the "low" spectral projection.
mid-high: Simply $\sin(k\,\boldsymbol{.})$ for $k = 21, 22, \ldots, 40 = N$.
high: All higher order modes ($k > N$).

Mapping a box $\boldsymbol{B = b + L(R_1)}$, where $b = C + R_2 + H$.

Consider: **a map** $\boldsymbol{f} : B \to \mathit{somewhere}$, with $f(0) = 0$,
for every x **a bound** $\boldsymbol{F(x)}$ on $Df(B)x$ (a convex set containing ...)
the **"corners"** $\boldsymbol{b + w_i}$ of B , where $w_1, w_2, \ldots, w_m$ are the corners of $L(R_1)$.

Bound on f from bound on Df: By convexity,

$$\boldsymbol{f(x) = \int_0^1 Df(tx)x \;\in\; F(x)\,, \qquad \forall x \in B\,.}$$

Convex combination of corners: Every $x \in B$ admits a unique representation

$$x = \xi + \sum_i s_i w_i\,, \qquad \xi \in b\,, \quad s_i \in [0,1]\,, \quad \sum_i s_i = 1\,.$$

We have

$$\boldsymbol{f(x) = \int_0^1 dt\, Df(tx)x = \sum_i s_i \int_0^1 dt\, Df(tx)(\xi + w_i) \;\in\; \sum_i s_i F(b + w_i)\,.}$$

Notice: The bounds $\{\boldsymbol{F(b + w_i)}\}_{i=1}^{m}$ are sufficient to estimate $f(x)$ for arbitrary $x \in B$.

The End