Yang-Mills in 2 dimensions for U(N) and its large-N limit

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The classical EM field

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The most familiar gauge field is the *electromagnetic field*.
The EM field is described by a *potential* which is a 1-form A on four-dimensional spacetime M.
It acts on a point charge e with the force

$$\text{force} = e i_v F$$

where v is the *velocity* of the charge and

$$F = -dA$$

describes the *strength* of the electromagnetic field.

Non-abelian gauge theory

Interaction between quarks is governed by a 1-form potential field A, buts values are skew-hermitian 3×3 matrices.
The field strength is

$$F^A = dA + A \wedge A$$

Classical field configurations are extrema of the *Yang-Mills action*

$$S_{\mathrm{YM}}(A) = \frac{1}{2g^2} \int_M \langle F^A, F^A \rangle \, d\mathrm{vol}$$

where g is a constant of physical significance, and integration is with respect to a volume measure on M.

Non-abelian gauge theory: quantum functional integral

Quantizing the gauge field itself requires (in one approach) using a *functional integral* measure

$$\frac{1}{Z_g} e^{-S_{\mathrm{YM}}(A)} DA$$

and one wants to compute integrals of the form

$$\frac{1}{Z_g} \int_{\mathcal{A}} f(A) e^{-S_{\mathrm{YM}}(A)} DA$$

for functions f of interest on the infinite-dimensional space $\mathcal{A}$.

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for functions f of interest on the infinite-dimensional space $\mathcal{A}$. Typical functions of interest are products of Wilson loop variables.

Simplifying the quartic

Now

$$S_{\mathrm{YM}}(A) \simeq \|dA + A \wedge A\|^2$$

which is *quartic* in A, and so

$$\frac{1}{Z_g} e^{-S_{\mathrm{YM}}(A)} DA$$

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Fortunately, we work only with $\mathcal{A}$ quotiented by a group $\mathcal{G}_o$ of symmetries (gauge transformations)
and this leads to a simplification in two dimensions.

YM on $\mathbb{R}^2$ is Gaussian

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Then

$$F^A = dA + \underbrace{A \wedge A}_{0} = dA = -\underbrace{\partial_y A_x}_{f^A} dx \wedge dy$$

YM on $\mathbb{R}^2$ is Gaussian

This makes out functional integral measure have a very convenient appearance:

$$\frac{1}{Z_g} e^{-\frac{1}{2g^2}\|f\|^2_{L^2}} Df$$

Gaussian measure in infinite dimensions

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Briefly, you take $\mathbb{R}$ with Gaussian measure $(2\pi)^{-1/2}e^{-x^2/2}dx$ and take an infinite product to obtain a probability measure on $\mathbb{R}^{\{1,2,3,\ldots\}}$.

YM on $\mathbb{R}^2$

To summarize, the Yang-Mills measure for gauge theory on $\mathbb{R}^2$ is rigorously meaningful and is Gaussian measure on the Hilbert space of functions

$$f : \mathbb{R}^2 \to L(G)$$

Technically it lives on a Hilbert-Schmidt completion of $L^2(\mathbb{R}^2) \otimes \mathrm{Lie}(G)$.
Note that the original connection form A is now a very rough object obtained by 'integrating' f.

Stochastic Geometry

Now consider a path

$$c : [0,1] \to \mathbb{R}^2 : t \mapsto \big(t, y(t)\big)$$

If A is a smooth connection on $\mathbb{R}^2$, parallel-transport along the path c is given by a path

$$[0,1] \to G : t \mapsto G : t \mapsto g_t$$

satisfying the differential equation

$$dg_t = -A\big(c'(t)\big) g_t dt$$

Now that A is stochastic, this can be reinterpreted as a Stratonovich *stochastic differential equation* (idea of L. Gross).

Holonomy and Wilson Loop Variables

If A is a connection and $c : [0,1] \to M$ a smooth loop, then $g_1 \in G$ is called the *holonomy* of A around c:

$$h_c(A) \stackrel{\text{def}}{=} g_1$$

Working with matrix groups G, we can form the trace

$$\mathrm{Tr}\big(h_c(A)\big)$$

which is a *Wilson loop variable*, as a function of the connection A.

$U(N)$ and heat kernel

We work with the *unitary group*

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$Q_t(x)$ is the *heat kernel* on the group $U(N)$. It solves

$$\frac{\partial Q_t(x)}{\partial t} = \frac{1}{2}\Delta Q_t(x)$$

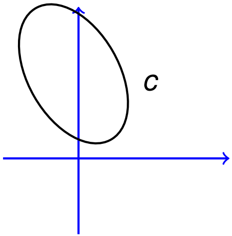
with initial condition

$$\lim_{t \downarrow 0} \int_{U(N)} f(x) Q_t(x)\, dx = f(I)$$

for every bounded continuous function f on $U(N)$; and dx is unit-mass Haar on $U(N)$.

Stochastic Holonomy

In quantum YM on the plane, each piecewise smooth simple closed loop c in $\mathbb{R}^2$ is associated with a *random variable* h_c with values in $U(N)$.



Loop expectation values notation

For a nice loop c, and a bounded measurable function f on $U(N)$ we have the expectation value

$$\mathbb{E}_N\left[f\left(h_c\right)\right]$$

We shall also write this as

$$\left\langle f\left(h_c\right)\right\rangle_N$$

or simply as

$$\left\langle f\left(h_c\right)\right\rangle$$

Conditions satisfied by stochastic holonomy

Conditions:

- $$\langle f(h_c) \rangle = \int_{U(N)} f(x) Q_{g^2 S}(x)\, dx$$

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  $$\langle \prod_{j=1}^{m} f_j(h_{c_j})\rangle = \prod_{j=1}^{m} \langle f_j(h_{c_j})\rangle$$

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(L. Gross, C. King. A.S.; Driver 1989)

Warm up: Laplacian of the Trace

Let

$$E_1, ..., E_D \tag{1}$$

be an orthonormal basis of the space of $N \times N$ hermitian matrices:

$$\mathrm{Tr}(E_a E_b) = \delta_{ab}, \tag{2}$$

where δ_{ab} is 1 if $a = b$, and 0 otherwise.

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$$\Delta \mathrm{Tr}_N(x) = \sum_{a=1}^{D} \partial_{iE_a} \mathrm{Tr}_N(xiE_a)$$

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The normalized trace

We will work with the normalized trace:

$$\mathrm{Tr}_N = \frac{1}{N}\mathrm{Tr}$$

Then

$$\mathrm{Tr}_N(I) = 1$$

Working out a Wilson loop expectation value

Recall

$$\langle \mathrm{Tr}_N h_c \rangle = \int_{U(N)} \mathrm{Tr}_N(x) Q_{g^2 S}(x)\, dx$$

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Using the heat kernel property:

$$\frac{\partial \langle \mathrm{Tr}_N h_c \rangle}{\partial S} = \frac{g^2}{2} \int \mathrm{Tr}_N(x) \Delta_x Q_{g^2 S}(x)\, dx$$

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Integrating by parts:

$$\frac{\partial \langle \mathrm{Tr}_N h_c \rangle}{\partial S} = \frac{g^2}{2} \int \Delta_x \mathrm{Tr}_N(x) Q_{g^2 S}(x)\, dx$$

One-loop expectation value differential equation

$$\frac{\partial \langle \mathrm{Tr}_N h_c \rangle}{\partial S} = \frac{g^2}{2} \int \underbrace{\Delta_x \mathrm{Tr}_N(x)}_{-N\mathrm{Tr}_N(x)} Q_{g^2 S}(x)\, dx$$

Writing $\tilde{g}^2 = g^2 N$, we have

$$\frac{\partial \langle \mathrm{Tr}_N h_c \rangle}{\partial S} = -\frac{\tilde{g}^2}{2} \langle \mathrm{Tr}_N h(C) \rangle$$

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Also, the loop expectation value solves the differential equation

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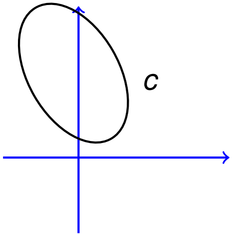
Hence

$$\langle \mathrm{Tr}_N h_c \rangle = e^{-\tilde{g}^2 S/2}. \quad (5)$$

Reminder: the simple loop c



h_c is a $U(N)$-valued random variable, and

$$\langle \mathrm{Tr}_N h_c \rangle = e^{-\tilde{g}^2 S/2}.$$

More moments

Next consider

$$W_N(c)_{\underline{k}} = \langle \mathrm{Tr}_N h_c^{k_1} \cdots \mathrm{Tr}_N h_c^{k_n} \rangle \tag{6}$$

for fixed $k = |\underline{k}|$, forming the components of a giant vector vector

$$\overrightarrow{W_N(c)}$$

in the vector space

$$V_k = \mathbb{C}^{\{\underline{k}:|\underline{k}|=k\}}$$

Moment differential equation

Theorem (F. Xu (1997); A.S. (2007))

If S is the area enclosed by the simple loop c then

$$\frac{\partial \overrightarrow{W_N(c)}}{\partial S} = -\frac{\tilde{g}^2}{2}\left[k\mathrm{I} + \mathrm{II} + \frac{2}{N^2}\mathrm{III}\right]\overrightarrow{W_N(c)}. \qquad (7)$$

Hence

$$\overrightarrow{W_N(c)} = e^{-\frac{\tilde{g}^2 S}{2}(k\mathrm{I}+\mathrm{II}+\frac{2}{N^2}\mathrm{III})}\mathbf{1}, \qquad (8)$$

where $\mathbf{1}$ *is the vector in* V_k *with all entries equal to* 1, *and* II *and* III *are linear operators (matrices).*

I, II, and III

$$\mathrm{I}f = f \tag{9}$$

and

$$\mathrm{II}f = \sum_{j=1}^{r} \mathrm{II}_j f, \tag{10}$$

where

$$(\mathrm{II}_j f)_{\underline{k}} = k_j \sum_{s=1}^{k_j - 1} f_{(k_1,\ldots,\not{k_j},s,k_j-s,\ldots,k_r)}, \tag{11}$$

and

$$(\mathrm{III}f)_{\underline{k}} = \sum_{1 \le l < m \le r} k_l k_m f_{(k_1,\ldots,\not{k_l},\ldots,\not{k_m},\ldots,k_r,k_l+k_m)} \tag{12}$$

The method of proof is a generalization of the method used for $\langle \mathrm{Tr} h_c \rangle$.

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T. Lévy (2008) provided a more insightful proof, along with many other related results, using Schur-Weyl duality of representations of S_n and $U(N)$.

Large-N limit

From

$$W_N(c) \stackrel{\text{def}}{=} \langle \mathrm{Tr}_N h_c \rangle = e^{-\tilde{g}^2 |S|/2}$$

we have

$$W_\infty(c) \stackrel{\text{def}}{=} \lim_{N\to\infty} W_N(c) = e^{-\tilde{g}^2 |S|/2}$$

exists.

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exists.
Indeed,

$$\overrightarrow{W_\infty(c)} = \lim_{N\to\infty} \overrightarrow{W_N(c)}$$

exists, on consulting the theorem mentioned before.

Remarkably the following factorization occurs:

$$\lim_{N\to\infty} \left\langle \prod_{j=1}^{r} \mathrm{Tr}_N\big(h_c^{k_j}\big) \right\rangle = \prod_{j=1}^{r} \lim_{N\to\infty} \langle \mathrm{Tr}_N\big(h_c^{k_j}\big)\rangle. \tag{13}$$

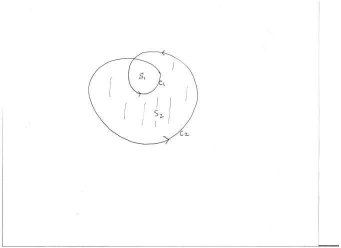
As a special case, for $k \in \{1, 2, ...\}$,

$$W_\infty(c^k) = e^{-k\frac{\tilde{g}^2}{2}S} P_k(\tilde{g}^2 S), \tag{14}$$

where $P_k(x)$ is an associated Laguerre polynomial of degree $k-1$.

Two loops in one

Inner loop c_1 encloses area S_1, and between c_1 and the outer loop c_2 lies area S_2.



Then

$$W_N(c_1 c_2) = e^{-\frac{\tilde{g}^2}{2}(S_2+2S_1)} \left(\cosh(\tilde{g}^2 S_1/N) - N \sinh(\tilde{g}^2 S_1/N)\right)$$

Singer's theory

I. M. Singer proposed that there is a 'universal bundle' over $\mathbb{R}^2$, and for each loop c in $\mathbb{R}^2$ there is a unitary operator U_c on an infinite-dimensional Hilbert space and

$$W_\infty(c) = \mathrm{Tr}_\infty U_c,$$

where Tr_∞ is a certain trace functional, and similar results hold for multiple loops and higher moments.

Stochastic Curvature

In quantum Yang-Mills on $\mathbb{R}^2$ to each subset $S \subset \mathbb{R}^2$ of finite area is associated a random matrix

$$F_N(S) \in \mathcal{H}_N$$

where $\mathcal{H}_N$ is the vector space of $N \times N$ hermitian matrices.

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where $\mathcal{H}_N$ is the vector space of $N \times N$ hermitian matrices. Informally,

$$iF_N(S) = \int_S \text{curvature} \in u(N) = i\mathcal{H}_N$$

Matrix-valued White Noise

More generally, $f \in L^2_{\mathrm{real}}(\mathbb{R}^2)$, a random $N \times N$ matrix

$$F_N(f)$$

satisfying the following conditions:

(i) $F_N(f)$ is a random hermitian matrix;

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(ii) $F_N(f)$ depends linearly on f;

(iii) for $f \neq 0$, the random variable $F_N(f)$ on $\mathcal{H}_N$ has density proportional to

$$e^{-N\frac{\mathrm{Tr}(T^2)}{2\|f\|^2}} \tag{15}$$

with T running over $\mathcal{H}_N$, the space of $N \times N$ Hermitian matrices.

Holonomy and Curvature

Given a curvature field F_N, holonomies h_c can be calculated by means of stochastic differential equations which mirror the equations of parallel-transport in differential geometry.

Algebraic Probability Space

An algebraic probability 'measure' on $\mathcal{A}$ is a linear map

$$\phi : \mathcal{A} \to \mathbb{C}$$

satisfying

$$\phi(1) = 1$$

and

$$\phi(aa^*) \geq 0 \quad \text{for all } a \in \mathcal{A}.$$

We will call $\mathcal{A}$, equipped with ϕ, an *algebraic probability space*.

Matrix Example

Take $\mathcal{A}$ to be the algebra of all $N \times N$ complex matrices, with the involution being the adjoint:

$$A \mapsto A^*$$

and the non-commutative probability measure being given by

$$\phi(A) = \mathrm{tr}_N(A) \stackrel{\mathrm{def}}{=} \frac{1}{N}\mathrm{tr}(A)$$

Random Matrix Example

Take $\mathcal{A}$ to be the algebra of all $N \times N$ matrices whose entries are complex-valued *random variables* on some probability space, with the involution being the adjoint, and the non-commutative probability measure being given by

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A special case of interest is

$$T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ T_{N1} & T_{N2} & \cdots & T_{NN} \end{bmatrix}$$

where

$$T_{ab} = S_{ab} + iA_{ab}$$

and the S_{ab}, A_{cd} are jointly Gaussian variables.

The Large-N Limit

Wigner's celebrated *semi-circular law* implies

$$\lim_{N\to\infty} \phi\left(F_N(f)^{2p}\right) = \|f\|^{2p} \frac{1}{p+1}\binom{2p}{p} \tag{16}$$

Thus

$$F_\infty(f) \stackrel{\text{def}}{=} \lim_{N\to\infty} F_N(f)$$

is a *semi-circular element* in a suitable probability algebra.

Freeness of subalgebras

Consider subalgebras $\mathcal{A}_1, ..., \mathcal{A}_N$, all closed under $*$. These are said to be *free* relative to each other if

$$\phi(a_1....a_M) = 0$$

for any $a_1, ..., a_M \in \mathcal{A}$, each with $\phi(a_j) = 0$, and *consecutive a_j belong to distinct $\mathcal{A}_i$*.

Applying Vociulescu's theorem to Noise

Returning to the orthogonal vectors $f_1, ..., f_m \in L^2(\mathbb{R}^2)$,and the corresponding independent Gaussian hermitian matrices $F_N(f_j)$, a fundamental result of Voiculescu implies:

$$(F_N(f_1), ..., F_N(f_m)) \xrightarrow{\mathrm{d}} (f_1', ..., f_m')$$

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Applying Vociulescu's theorem to Noise

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$$(F_N(f_1), ..., F_N(f_m)) \xrightarrow{\mathrm{d}} (f_1', ..., f_m')$$

where $f_1', ..., f_m'$ are mutually free elements in some algebraic probability space, and

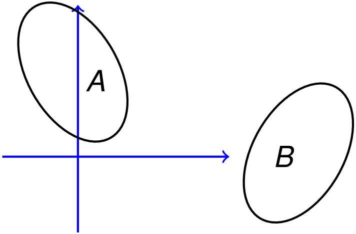
> each f_j' is *semicircular* with radius $2\|f_j\|$ (if f_j is 0 then f_j' is 0).

Free limit of the curvature

Intepreting the preceding result in the context of stochastic curvature shows:

Theorem

The stochastic curvature field $F_N(\cdot)$ converges in distribution to a free white noise process on the plane.



$F(1_A)$ and $F(1_B)$ are free

Figure: Free noise

Objectives/Challenges

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- Connect to Rajeev's QHD Grassmanian phase space for large-N QCD

Thank you! Obrigado!