Stochastic wave equation model for heat-flow in non-equilibrium statistical mechanics

We consider a one-dimensional non-linear stochastic wave equation system modeling heat flow between thermal reservoirs at different temperatures. We will briefly review the problem of solving these equations in Sobolev spaces of low regularity. The system with ultraviolet cutoffs has, for each cutoff, a unique invariant measure exhibiting steady-state heat flow We provide estimates on the field covariances with respect to the invariant measures which are uniform in the cutoffs.

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1.1 Introduction

1.1.1 Review

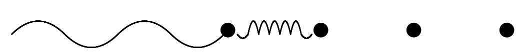
- (J.-P. Eckmann, C.-A. Pillet, L. Rey-Bellet, *Commun.Math.Phys* **201** (1999)

$$H \equiv H_B + H_S + H_I \tag{1.1}$$

$$\begin{aligned} H_B &= H(\phi_L, \pi_L) + H(\phi_R, \pi_R) \\ H(\phi, \pi) &= \frac{1}{2} \int (|\nabla(\phi)|^2 + |\pi|^2)\, dx \end{aligned} \tag{1.2}$$

$$H_S(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{2} + U^{(1)}(q_i) + \sum_{i=1}^{n-1} U^{(2)}(q_i - q_{i+1}) \quad (1.3)$$

$$H_I = q_1 \int \nabla\phi_L(x)\rho_L(x)\,dx + q_n \int \nabla\phi_R(x)\rho_R(x)\,dx \quad (1.4)$$

- 

- Assume Fourier transforms of ρ's have a certain rational function form.
- Give left and right fields Gaussian distributed random initial conditions (temperatures T_L and T_R).
- Integrate out bath field variables to get:

$$\begin{aligned} \frac{dq_i}{dt} &= p_i, i = 1, 2, .., n && (1.5) \\ \frac{dp_i}{dt} &= -\nabla_i V(q) + \delta_{i,1} r_L + \delta_{i,n} r_n,\ i = 1, 2, ..., n \\ dr_\cdot &= -(r_\cdot + p_\cdot) + \sqrt{2T_\cdot}\, d\omega_\cdot \ \ \cdot = L = 1 \text{ or } R = n \end{aligned}$$

- Assume $U^{(1)}(q) \propto q^{k_1}$, $U^{(2)}(q) \propto q^{k_2}$, $q \to \infty$, $k_2 \geq k_1$.

Theorem 1.1.1 *(E.P.R.-B.; R.-B., T, Commun. Math. Phys.* **225** *(2002).) Process $x(t) = (p, q, r)(t)$ has unique invariant measure μ (with smooth density),*

$$P_t(x, \cdot) \to \mu \text{ exponentially fast.} \quad (1.6)$$

In the stationary state, there is heat flow, $T_L \neq T_R$ and entropy production, also Gallavotti-Cohen relation. If $T_L = T_R$ get $\mu =$ Gibbs state.

- Describes steady state heat flow from one bath to the other.

1.2 Stochastic non-linear wave equation

- What if number of degrees of freedom increases?
- Fourier or Ohm's law?
- Equations for a field medium.

$$\begin{aligned} \partial_t \phi(x,t) &= \pi(x,t) \\ \partial_t \pi(x,t) &= (\partial_x^2 - 1)\phi(x,t) - \mu\phi^3(x,t) - r(t)\alpha(x) \\ dr_i(t) &= -\left(r_i(t) - \langle \alpha_i, \pi(t) \rangle\right) dt + \sqrt{2T_i}\, d\omega_i(t) \quad i = 1,2 \end{aligned} \tag{1.7}$$

- $\phi(x,t)$ is scalar field, $x \in [0, 2\pi]$. $\pi(x,t)$ is conjugate momentum.
- $r(t) = (r_L(t), r_R(t))$ is an artifact of the baths.
- $\alpha(x) = (\alpha_L(x), \alpha_R(x))$ are *fixed* functions coupling the field to the baths.
- $T = (T_L, T_R)$ are left and right temperatures.
- $d\omega = (d\omega_L, d\omega_R)$ are driving Brownian motion terms.

1.3 Local and Global existence for Stochastic Wave Equation

(R.-B., T.-*Stochastic Processes and their Applications* **115** (2005). Based on J. Bourgain, Global Solutions non-linear Schr0̈dinger Equation, *AMS Colloquium Publications*, **46**, (1999).)

Notation

Set

$$\Phi(\omega, t) = (u, r)(x, t) = (\phi + \frac{i}{\sqrt{-\partial_x^2 + 1}}\pi, r)(x, t). \quad (1.8)$$

Set

$$\mathcal{D}_R(\beta, t) \equiv \Big\{\mathbf{u}(\cdot) \in \mathcal{C}([0, t], H^s) \mid \|\mathbf{u}(0)\|_{H^s} \leq \beta \\ \text{and } \sup_{t' \leq t} \|\mathbf{u}(t')\|_{H^s} \leq R\beta\Big\}, \quad (1.9)$$

and let $\mathcal{F}_R(\beta, t)$ be the (probabilistic) event that the equations of motion have a unique *strong* solution in $\mathcal{D}_R(\beta, t)$.

Proposition 1.3.1 $\{$*Local Existence* $\}$*Assume* $\frac{1}{6} < s < 1$. *There exist constants* c_1, c_2, c_3 *and* C *such that if* $\Phi(0)$ *satisfies* $\||\Phi(0)\|_{H^s} \leq \beta$, $R > 3c_3$ *and* $t \leq c_1/(R^2\beta^2)$, *then*

$$P\{\mathcal{F}_R(\beta, t)\} \geq 1 - C \exp\left(-\frac{c_2 R^2 \beta^2}{t(1+t)^2}\right). \quad (1.10)$$

Corollary 1.3.2 *For* $s > \frac{1}{6}$, *local existence of the solution* $\Phi(\cdot)$ *in* H^s *holds almost surely,*

$$P\{\cup_n \mathcal{F}_R(\beta, t/n)\} = 1. \quad (1.11)$$

Proposition 1.3.3 *(Global Existence) Let* $s \geq \frac{1}{3}$. *Let* $\beta = \|\Phi(0)\|_{H^s}$. *There exist constants,* c, C, θ *and* $N_1 = N_1(\beta, t)$, *such that for any time* t, *and* $N \geq N_1$,

$$P\left\{\sup_{t' \leq t} \|\Phi(t')\|_{H^s} > \beta N\right\} \leq C \exp\left(-\frac{cN^{2\theta}}{t(1+t)^2}\right) \tag{1.12}$$

1.4 Results on the Linear Problem (Spectral Theory)

with Y. Wang (*Contemporary Math* **447**, (2007))

- Let G be generator of linear equation,

$$G \equiv \begin{pmatrix} 0 & 1 & O & 0 \\ \partial_x^2 - 1 & 0 & -\alpha_L & -\alpha_R \\ 0 & \langle \alpha_L | & -1 & 0 \\ 0 & \langle \alpha_R & 0 & -1 \end{pmatrix} \tag{1.13}$$

- Linear system (no cut-offs), $T_L = \neq T_R$ (or $T_L = T_R$) is weakly ergodic, $(\phi(f), \pi(g), r)(t)$ converge in measure, $t \to \infty$. Idea: Ornstein-Uhlenbeck Process

$$\begin{aligned} \Phi(t) &= \int_0^t e^{(t-s)G} \begin{pmatrix} 0 \\ 0 \\ \sqrt{T}\, d\omega(s) \end{pmatrix} + e^{tG}\Phi_0 \\ &\to \int_{-\infty}^0 e^{-sG} \begin{pmatrix} 0 \\ 0 \\ \sqrt{2T}\, d\omega(s) \end{pmatrix}. \end{aligned}$$

- Latter expression is for a field Φ which has distribution ν, the invariant measure.

- Spectral information for G. G has compact resolvent, spectrum tending to $\pm\infty$, with nearly degenerate eigenvalues for each $n \in \mathbf{Z}$.

- Linear system is formally hypoelliptic (but need all Fourier coefficients of *both* coupling functions α to be non-zero).

- Linear system (non-equilibrium $T_L \neq T_R$) has unique invariant measure with sample field configurations as regular as equilibrium case, e.g., $\phi(x)$ is Hölder continuous with index $1/2^-$. (Brownian motion- or Brownian bridge-like).

  Moreover stationary field is Hölder continuous in x with index $1/2 - 0$.

1.4.1 Ultraviolet Cutoff Convergence

Let $\Phi_M(\cdot)$ be solution to ultraviolet cutoff equations.

$$\mathcal{G}_R(\beta, t) \equiv \{\mathbf{u}(\cdot),\ \mathbf{u}_M(\cdot) \in \mathcal{D}_R(\beta, t) \text{ for each } M\} \quad (1.14)$$

Proposition 1.4.1 *Fix $s > 1/3$, a time $t > 0$, and $s_o > s$. Then $\{\Phi_M(\cdot)\}$ converges strongly to $\Phi(\cdot)$ in H^s uniformly on $\mathcal{G}_R(\beta, t) \cap \{\mathbf{u} \mid \|\mathbf{u}(0)\|_{H^{s_o}} \leq \beta\}$. Also, let*

$$S^t f(\Phi) \equiv E_\Phi[f(\Phi(t))],\ S_M^t f(\Phi) \equiv E_\Phi[f(\Phi_M(t))]; \quad (1.15)$$

Then $S_M^t f(\Phi) \to S^t f(\Phi)$, for $M \to \infty$ uniformly in Φ, $\|\Phi\|_{H^s} \leq \beta$.

1.4.2 Equilibrium Invariant Measure, $T_L = T_R$

$$
\begin{aligned}
d\nu = & \\
& Z^{-1} \exp\left(-\frac{1}{2T}\int\left(|\partial_x \phi(x)|^2 + |\phi(x)|^2 + \frac{\mu}{2}|\phi(x)|^4 + |\pi(x)|^2\right)\, dx\right) \\
& \qquad \times \exp\left(-\frac{1}{2T}r^2\right)\, dr \prod_{x\in[0,2\pi]} d\phi(x) d\pi(x)\,. \qquad (1.16)
\end{aligned}
$$

Proposition 1.4.2 *(Equilibrium case, $T_L = T_R$) The measure ν is invariant with respect to the semigroup S^t in the sense that for $f \in \bar{\mathcal{X}}$, (suitable function space)*

$$
\int S^t f\, d\nu = \int f\, d\nu. \qquad (1.17)
$$

1.5 Recent Results (with Y. Wang)

- Take non-linearity bounded, Lipschitz, $\hat{\alpha}(n) \sim n^\theta$, $-1/2 < \theta < 1/2$.
- Systems with ultraviolet cut-offs have unique invariant measures $\{\mu_M\}$. Let $\hat{\Phi}_M(n)$ denote the n Fourier mode of the (stationary) field Φ_M with respect to the M^{th} ultraviolet cutoff.

  Proposition 1.5.1 *The variances of $\hat{\Phi}_M(n)$ are, for each fixed n, uniformly bounded in the ultraviolet cut-off.*

  $$E_{\mu_M}\left[|\hat{\Phi}_M(n)|^2_{H^1}\right] \leq C(1+n^2)^{1-\theta}$$

  and

  $$E_{\mu_M}\left[|\Phi_M|^2_{H^\theta}\right] \leq C$$

  uniformly in M. Invariant measures $\{\nu_M\}$ have weak limit points supported on a space of distributions.*

- One expects these variances to be uniformly bounded in n as well (as in the linear case) (approximate equipartition of energy), If uniform (in n) bound then ϕ is in $H^{1/2}$ and is nearly continuous.

- Violin strings and all that. Critical phenomena?