Stochastic reversible deformation of dynamical systems

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I. Stochastic processes and reversibility

Wiener process associated with $H = -(\hbar^2/2)\Delta$. Archetype of irreversibility.
Origin of irreversibility in Stochastic Analysis: historic, not scientific.

Kolmogorov and probabilistic initial value problem.

Markov process $X_t \in \Lambda,\ t \in I$, on $(\Omega, \mathfrak{a}, P)$, $\mathcal{P}_t = \sigma\{X_\tau, \tau \le t\} \subset \mathfrak{a}$.

Usual formulation (time asymmetric) of X Markovian:

$$E\Big[f(X_t) \,|\, X(t_{-1}), X(t_{-2}), \ldots, X(t_{-n})\Big] \qquad s < t_{-n} < \ldots < t_{-2} < t_{-1} \le t$$
$$= E\Big[f(X_t) \,|\, X(t_{-1})\Big] \qquad \text{i.e. "forget the past } \mathcal{P}_{t_{-1}} \text{ of } t_{-1}\text{"}$$

Building block of construction of measure: Kolmogorov's transition probability

$$P(s, X_s, t, B) = P(X_t \in B \,|\, X_s), \quad s \le t, \quad B \text{ Borelian.}$$

When X_s given by a distribution P_s , $s \le \tau_1 \le \tau_2 \le \ldots < u$

$$P(X_{\tau_1} \in B_1, \ldots, X_{\tau_n} \in B_n) = \int P_s(dx)\, P(s, x, \tau_1, B_1) \ldots P(\tau_{n-1}, x_{n-1}, \tau_n, B_n)$$

Finite-dimensional distribution of X.

Probabilistic analogue of *initial* value problem for classical dynamics.

Smooth paths space $\Omega_{x,s} = \left\{\omega \in C^2([s,u];\ \mathbb{R}) \text{ s.t. } \omega(s) = x\right\}$.

But Markov property is *time-symmetric*. If

$$\mathcal{F}_t = \sigma\{X_\tau,\ \tau \geq t\}, \quad A \in \mathcal{P}_t, \quad B \in \mathcal{F}_t,$$

$$P(AB\,|\,\mathcal{P}_t \cap \mathcal{F}_t) = P(A\,|\,\mathcal{P}_t \cap \mathcal{F}_t) \cdot P(B\,|\,\mathcal{P}_t \cap \mathcal{F}_t)$$

$\Rightarrow$ backward Markovian evolution from a *final* distribution $P_u(dz)$:

$$P(X_{\tau_1} \in B_1, \ldots, X_{\tau_n} \in B_n) = \int P^*(\tau_1, B_1, \tau_2, x_2) \ldots P^*(\tau_n, B_n, u, z) P_u(dz)$$

P^* = backward transition probability.

$$E\Big[f(X_t)\,|\,X(t_1), X(t_2), \dots, X(t_n)\Big] = E\Big[f(X_t)\,|\,X(t_1)\Big], \quad s < t < t_1 < t_2 < \dots < u$$

"forget the future $\mathcal{F}_{t_1}$ of $X(t_1)$" Also time asymmetric!

Probabilistic counterpart of final value problem for classical dynamical systems. Smooth path space $\Omega^{u,z}$.

What is probabilistic counterpart of boundary value variational problem with

$$\Omega^{z,u}_{x,s} = \Big\{\omega \in C^2([s,u];\, \mathbb{R}] \text{ s.t. } \omega(s) = x, \omega(u) = z\Big\} \;?$$

Probabilistic idea: Forget the past of any s_1 *and* the future of any t_1:

$$E\Big[f(Z_t)\,|\,\mathcal{P}_{s_1} \cup \mathcal{F}_{t_1}\Big] = E\Big[f(Z_t)\,|\,Z(s_1), Z(t_1)\Big] \qquad \forall\, s \le s_1 < t < t_1 \le u$$

"Bernstein property" (1985–6, or "local Markov" or "two-sided Markov" or "Reciprocal", 1932): more general than Markov (OK for $Z(s_1)$ or $Z(t_1)$ fixed).

Data: Initial *and* final probabilities $P_s(dx)$, $P_u(dz)$.

> How to construct unique process Z_t, $t \in [s, u]$
> from given joint probability measure $M(dx, dz)$?

1) Build 3 points counterpart of transition probability $Q(s, x; t, B; u, z)$
   s.t. if Z_u fixed Q behaves like forward transition probability $P(s, x, t, B)$
   and, if Z_s fixed, as a backward transition $P^*(t, B, u, z)$.
2) Use $M(dx, dz)$ instead of $P_s(dx)$, $P_u(dz)$
   to write finite-dimensional distributions of Z_t, $\forall\, t \in [s, u]$.

Then $\exists!$ Bernstein Z_t, $t \in [s, u]$, generally not Markovian.

Difficulty: For given (P_s, P_u) $\exists$ many M (marginals not sufficient).

"Solution": $\exists!$ M for given (P_s, P_u), denoted by M_m, s.t. $Z.$ is also Markovian.

Let us consider a class including Wiener process. Then

$$M_m(dx, dz) = \eta_s^*(x)\, h(x, u-s, z)\, \eta_u(z)\, dx\, dz, \qquad \eta_s^*,\ \eta_u > 0$$

$$h(x, u-s, z) = \left(e^{-(u-s)H/\hbar}\right)(x, z) \qquad \text{on } L^2(\mathbb{R}^n),\ \hbar > 0.$$

$$H = -\frac{\hbar^2}{2}\Delta + V, \qquad \text{s.t. } h > 0, \text{ jointly continuous, } V \in \text{ Kato class.}$$

Then, finite dimensional distributions of Z_t, $t \in [s, u]$:

$$P(Z_{t_1} \in B_1, \ldots, Z_{t_n} \in B_n) = \int \eta_s^*(x)\, h(x, t_1 - s, B_1) \ldots h(B_n, u - t_n, z)\, \eta_u(z)\, dx\, dz$$

Marginals of M_m determine η_s^*, η_u, given $P_s(dx) = p_s(x)\, dx$, $P_u(dz) = p_u(z)\, dz$:

$$\begin{cases} \eta_s^*(x) \displaystyle\int h(x, u - s, z)\, \eta_u(z)\, dz = p_s(x) \\ \eta_u(z) \displaystyle\int \eta_s^*(x)\, h(x, u - s, z)\, dx = p_u(z) \end{cases} \qquad \text{Nonlinear integral eq. for } (\eta_s^*, \eta_u).$$

Theorem A. Beurling (1960, Ann. Math):

h as before, $\exists!$ positive (not necessarily integrable) solutions $\eta_s^*,\ \eta_u$ if p_s and p_u are strictly positive.

Ex:
$H = H_0,\ p_s = \delta_x,\ p_u = h_0(x, u - s, z)$ ("free" Gaussian kernel). Then $\eta_s^*(x) = \delta_x$, $\eta_u(z) = 1$ and $Z_t = W_t$. The Brownian irreversibility is only a special scenario of free ($V = 0$) evolution. This Brownian will allow, however, to construct many other free (reversible) diffusions.

Properties of Bernstein Markovian processes:

1) $P(Z_t \in A) = \int_A \eta^*(q,t)\,\eta(q,t)\,dx, \quad t \in [s,u]$

$$\begin{cases} -\hbar \dfrac{\partial \eta^*}{\partial t} = H\eta^* \\ \eta^*(x,s) = \eta^*_s(x) \end{cases} \qquad \begin{cases} \hbar \dfrac{\partial \eta}{\partial t} = H\eta \\ \eta(x,u) = \eta_u(x) \end{cases}$$

Smooth $\eta, \eta^* > 0$

H as before

2) $\hat{Z}(t) = Z(u+s-t)$, $t \in [s,u]$ well defined. Built-in reversibility !
$\hat{Z}.$ has boundary probabilities at time s and u permuted with respect to the ones of $Z.$

3) Infinitesimal coefficients (1 dim, for $H = -(\hbar^2/2)\Delta + V$):

Forward	$\mathcal{P}_t$ – SDE	$dZ_t = \hbar \dfrac{\nabla \eta}{\eta}(Z_t, t)\, dt + \hbar^{1/2} dW_t$
Backward	$\mathcal{F}_t$	$d_* Z_t = -\hbar \dfrac{\nabla \eta^*}{\eta^*}(Z_t, t)\, dt + \hbar^{1/2} d_* W_t^*$

(W^* = Backward Wiener)

4) Infinitesimal generators:

$$D_t f(Z(t),t) = \lim_{\Delta t \to 0} E_t \left[\frac{f(Z(t+\Delta t), t+\Delta t) - f(Z(t),t)}{\Delta t} \right]$$

$$D_t^* f(Z(t),t) = \lim_{\Delta t \to 0} E_t \left[\frac{f(Z(t),t) - f(Z(t-\Delta t), t-\Delta t)}{\Delta t} \right]$$

where E_t = conditional expectation (given $Z(t)$)

$$D_t = \frac{\partial}{\partial t} + \hbar \frac{\nabla \eta}{\eta} \nabla + \frac{\hbar}{2} \Delta \qquad ; \qquad D_t^* = \frac{\partial}{\partial t} - \hbar \frac{\nabla \eta^*}{\eta^*} \nabla - \frac{\hbar}{2} \Delta$$

For $D_t Z = B(Z,t) = \hbar \frac{\nabla \eta}{\eta}(Z,t)$, $D_t^* Z = B^*(Z,t)$ ("drifts")

$B^*(Z,t) = B(Z,t) - \hbar \nabla \log(\eta^* \eta)(Z,t)$

<u>NB</u>: Only in stationary case $B^* = -B$ (as when $\hbar = 0$).

Ex: Wiener (starting from x at time s)

$$\eta(q,t) = 1 \Rightarrow B = 0$$
$$\eta^*(q,t) = h_0(x, t-s, q) \Rightarrow B_*(q,t) = -\hbar\nabla \log h_0(x, t-s, q)$$

Deformations of constants under dynamics:

$$f(Z,t) \quad \text{s.t.} \quad D_t f = 0 \;\; \text{is } \mathcal{P}_t\text{-martingale}$$
$$g(Z,t) \quad \text{s.t.} \quad D_t^* g = 0 \;\; \text{is } \mathcal{F}_t\text{-martingale}$$

II. Reversible stochastic deformation of dynamical systems

Singular $\lim\limits_{\hbar \to 0} \quad \sim \quad$ classical limit of QM.

Above framework = stochastic deformation of such limiting classical system.
Let $H(x, p)$ its Hamiltonian, $L(x, \dot{x})$ its Lagrangian.

Claim: All classical Theorems needed to analyse this classical dynamical system are deformable a.s. along Bernstein processes $t \longmapsto Z_t$.

A selection of examples:

1) Stationary case, $H = -(\hbar^2/2)\Delta + V(x)$, State space $\Lambda =$ interval of $\mathbb{R}$

$\eta(x,t) = g_\alpha(x)e^{-\alpha t/\hbar}$, $\eta^*(x,t) = g_\alpha(x)e^{\alpha t/\hbar}$, $\alpha =$ cst.
$p_s(x) = p_u(x) = |g_\alpha(x)|^2$ invariant probability density.

Deformation of Lagrangian-Newtonian approach

$$S_L(x,t) = E_{xt}\int_t^{\hat{\tau}} L\Big(X(\tau), D_\tau X(\tau)\Big)d\tau \equiv S_L[X(\cdot)], \qquad E[\hat{\tau}] < \infty$$

(upper integration limit $\hat{\tau}$ random when Λ bounded)

$$\mathcal{D}_{S_L} = \Big\{X_\cdot \text{ solving } \mathcal{P}_t\text{-SDE, fixed diffusion coef.,} \quad \text{smooth } D_\tau X \text{ to be determined}\Big\}$$

Def: Critical diffusions Z of $S_L[X(\cdot)]$: $\delta S_L = E_{xt}[\mathbf{\nabla} S_L[Z](\delta Z)] = 0$, "$\forall\, \delta Z$",

$$\mathbf{\nabla} S_L[Z](\delta Z) = \lim_{\varepsilon \to 0} \frac{S_L[Z + \varepsilon \delta Z] - S_L[Z]}{\varepsilon} \qquad \text{a.s. Gâteaux derivative}$$

$\delta Z(\cdot) \in$ Cameron-Martin (Hilbert) space $\mathcal{H}_{\mathrm{CM}}$ with $\langle \varphi, \psi \rangle = \int_0^\infty \dot{\varphi}(\tau)\, \dot{\psi}(\tau)\, d\tau$

$$E_{xt} \int_t^{\hat{\tau}} \left(\frac{\partial L}{\partial X} \delta X + \frac{\partial L}{\partial D_\tau X} D_\tau \delta X \right) d\tau = 0 \qquad \forall\, \delta X \in \mathcal{H}_{\mathrm{CM}}$$

Integration by parts w.r.t. D_τ (Itô's formula for $\forall\, \delta X \in \mathcal{H}_{\mathrm{CM}}$)

$$D_\tau \frac{\partial L}{\partial D_\tau Z} - \frac{\partial L}{\partial Z} = 0 \qquad \text{a.s. deformed Euler-Lagrange}$$

Elementary $H \Rightarrow L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + V(q)$ so

Theorem

The critical diffusions of S_L in $\mathcal{D}_{S_L}$, for this L, are Bernstein Markov processes solving a.s. $D_\tau D_\tau Z(\tau) = \nabla V(Z(\tau))$

(Deformed Newton).

2) A variational principle with (non holonomic) constraint: 1d Maupertuis Principle

$Z(\cdot) : [s, u] \longrightarrow [0, y] = \Lambda \subset \mathbb{R}$
$H(V)$ time independent. Stationary case: $p_s = p_u$.
Drifts time independent: $B^* = -B$
and satisfy deformed energy conservation constraint:

$$\frac{1}{2}|B_\alpha|^2(Z(\tau)) + \frac{\hbar}{2}\nabla B_\alpha(Z(\tau)) - V(Z(\tau)) = \alpha \qquad (*)$$

Theorem

The critical diffusions of S_L are also critical for the deformed reduced action

$$E_{xt}\int_t^{\tau^y} B_\alpha(Z(\tau)) \circ dZ(\tau) \qquad \circ \text{ Stratonovich differential}$$

in the (narrower) class of diffusions $Z_\alpha(\cdot)$ as above satisfying $(*)$, and

$$\tau^y = \inf\Big\{\tau \geq t : Z(\tau) = y \,|\, Z(t) = x\Big\}.$$

Idea:

1)

$$S_L[Z(\cdot)] = E_{xt} \int_t^{\tau^y} B(Z(\tau)) \circ dZ(\tau) - \int_t^{\tau^y} \underbrace{\left(\frac{1}{2}|B|^2 + \frac{\hbar}{2}\nabla B - V\right)}_{= \alpha \text{ on } Z_\alpha(\cdot)}(Z(\tau))\, d\tau$$

2) Not any $g_\alpha(x)$ qualifies as probability density of critical diffusions $Z(\cdot)$ since both $\partial\Lambda$ are attainable. But

$$g_\alpha^+(x) = g_\alpha(x) \int_0^x g_\alpha^{-2}(\xi)\, d\xi \quad \Rightarrow \quad q_\alpha^+(x) = \frac{g_\alpha^+(x)}{g_\alpha(x)} \frac{g_\alpha(y)}{g_\alpha^+(y)} > 0$$

$$D_\tau q_\alpha^+(Z_\alpha(\tau)) = 0, \qquad q_\alpha^+(0) = 0, \quad q_\alpha^+(y) = 1 \quad (\mathcal{P}_\tau\text{-Martingale})$$

Doob's transformation of $Z(\tau)$ via positive martingale

$q^+_\alpha \to Z^+_\alpha(t),\ t \in [s,u],\ B^+_\alpha(x) = \hbar \dfrac{\nabla g^+_\alpha}{g^+_\alpha}(x),$

Z^+_α cannot reach origin anymore but solves same a.s. Newton equation.

3)

In addition Z^+_α solves a.s. $\begin{cases} D_\tau m(Z^+_\alpha(\tau)) = -1\,, & \tau \in [t, \tau^y] \\ m(y) = 0 & \text{Not Dirichlet problem! } (m(\partial\Lambda) \neq 0) \end{cases}$

$m(x) = E_{xt}[\tau^y] - t$

Deformation of characteristics

Def: Deformed Hamilton *characteristic function*:

$$W_\alpha^+(x) = -\hbar \log g_\alpha^+(x) \qquad x \in \Lambda$$

solves *reduced Hamilton-Jacobi-Bellman* equation:

$$\frac{1}{2}|\nabla W_\alpha^+|^2(x) - \frac{\hbar}{2}\Delta W_\alpha^+(x) - V(x) = \alpha \tag{HJB}$$

Since $B_\alpha^+ = -\nabla W_\alpha^+$, (HJB) $\Leftrightarrow$ Deformed energy conservation

$$D_t W_\alpha^+(Z_\alpha^+(t)) = -|B_\alpha^+(Z_\alpha^+(t))|^2 - \frac{\hbar}{2}\nabla B_\alpha^+(Z_\alpha^+(t)) \ \Rightarrow \qquad \text{(Cf Stratonovich / Itô)}$$

$$W_\alpha^+(x) = E_{x,t}\int_t^{\tau^y} B_\alpha^+(Z_\alpha^+(\tau)) \circ dZ_\alpha^+(\tau) \qquad \text{Deformed reduced action}$$

As classically $\qquad \nabla(\text{HJB}) \longrightarrow D_t D_t Z_\alpha^+(t) = \nabla V(Z_\alpha^+(t))$

Same method for Z_α^- conditioned on reaching only lower border 0.

3) Time dependent variational principle $\Lambda = \mathbb{R}^n$

$$H = -\frac{\hbar^2}{2}\Delta + V \qquad \text{Any } p_s(x) \text{ and } p_u(x) \qquad \text{(Beurling)}$$

$$S_L(x,t) = E_{xt}\int_t^u L\Big(X(\tau), D_\tau X(\tau)\Big)\,d\tau + E_{xt}S_L(X(u),u)$$

$\Lambda = \mathbb{R}^n$, no need of random time

$$\text{Critical diffusions:} \quad \begin{cases} D_\tau \dfrac{\partial L}{\partial D_\tau Z} - \dfrac{\partial L}{\partial Z} = D_\tau D_\tau Z(\tau) - \nabla V(Z(\tau)) = 0 & \text{a.s.} \\ \text{BC} : Z(t) = x, \qquad DZ(u) = -\nabla S_L(Z(u),u) \end{cases}$$

S_L solves time-dependent *Hamilton-Jacobi-Bellman equation*

$$\frac{\partial S_L}{\partial t} - \frac{1}{2}|\nabla S_L|^2 + \frac{\hbar}{2}\Delta S_L + V = 0 \tag{HJB}$$

$$DZ(\tau) = -\nabla S_L(Z(\tau),\tau) \qquad \text{true } \forall\, t \leq \tau \leq u$$

$$\nabla(\text{HJB}) \quad \text{is} \quad D_t D_t Z(t) = \nabla V(Z(t))$$

There is also a stochastic deformation of *Nœther's Theorem*, for S_L:

$D_t(pX + hT - \phi)(Z(t),t) = 0$ for X, T, ϕ = infinitesimal generator coefficients of symmetry group of heat equation, and $p = -\nabla S_L$, $h = -\partial_t S_L$.
For recent geometric approach, cf P. Lescot, J.-C. Z.

It produces a collection of $\mathcal{P}_t$ ($\mathcal{F}_t$) martingales of critical $Z(\tau)$.

Stochastic dynamical system reinterpretation of martingales of diffusions.
Ex: Wiener's martingales (cf Chung, J.-C. Z. p181–185) for $V = 0$

4) $\Lambda = n$-dim. Riemannian manifold. $d\mu(q) = \sqrt{g}\, dq\,, g = \det g_{ij}$

Framework preserved for huge class of H (i.e. Bernstein). Ex:

$$Hf(k) = U(k)f(k) - c\nabla F - \frac{1}{2}\Delta f - \int_{\mathbb{R}^n} \Big(f(k+y) - f(k) - y\nabla f(k) \cdot \mathbb{1}_{\{|y|\leq 1\}}\Big)\nu(dy)$$

$c, k \in \mathbb{R}^n$, ν Lévy measure on $\mathbb{R}^n \setminus \{0\}$ **NB:** H not symmetric !

Then two underlying heat equations involve H and its adjoint H^+.
(cf Privault, J.-C. Z.)

Generator $D_t =$ Integro-differential.

Stochastic Analysis can be entirely time-symmetrized and, but only after, be regarded as a (stochastic) dynamical systems theory.
Other interpretation of this program: Systematic Analysis of PDE with the (deformed) tools of ODE. Geometric flavour.

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