PROBLEM LIST

1. İnanç Baykur

Prove, or disprove by constructing counter-examples:

Problem 1 (Symplectic Poincare Conjecture)**.** *Any closed symplectic 4-manifold homeomorphic to CP^2 is diffeomorphic to it.*

This is true in the complex category. On the other hand, there is no closed 4-manifold that is known to admit a unique smooth structure, and 4-manifolds with small topology, such as CP^2, challenge existing construction techniques –and require further creativity.

Problem 2 (Symplectic Calabi-Yau Conjecture)**.** *Any closed symplectic 4-manifold with torsion canonical class is diffeomorphic to a K3 surface, Enriques surface, or a torus bundle over torus.*

This is true in the complex category (where the list is even shorter). While there is a complete classification for symplectic rational and ruled surfaces (i.e. when Kodaira dim is negative), only a homological classification is accomplished for symplectic Calabi-Yaus (i.e. when Kodaira dim is zero).

2. Shelly Harvey

The smooth knot concordance group C contains a subtle and interesting subgroup, T, the group of topologically slice knots. Tim Cochran, Shelly Harvey, and Peter Horn defined a filtration, called the bipolar filtration, on T, $\{T_n\}$, generalizing work of Tim Cochran and Bob Gompf on the positivity of a knot. This is a refinement of the n-solvable filtration. It was recently show by Jae Choon Cha and Min Hoon Kim that that the successive quotients T_n/T_{n+1} is non-trivial for all n. The proof that the topologically slice knots are not smoothly slice (or in T_{n+1}) required a subtly combination of L^2 rho invariants and d-invariants of p-fold branched covers for an infinite number of primes, p.

Problem 3. *Are there any other invariants from Heegaard Floer that obstruct a knot lying in T_{n+1}?*

Problem 4. *Does T_n/T_{n+1} have 2-torsion?*

Problem 5. *Does T_n/T_{n+1} have a Z^∞ summand?*

A satellite operator is a function from C to itself. We would like to know if there are any doubling operators (satellite operators with algebraic winding number zero) that are injective. This would give evidence that C has the structure of a fractal space. Here, the natural metric to study is the bipolar metric defined by Tim Cochran, Shelly Harvey, Mark Powell, and Aru Ray using kinky disks and gropes.

Problem 6. *Are there any injective doubling operators that are injective and contractions under the bipolar metric.*
More specifically, is the Whitehead doubling operator weakly injective (i.e. Is Wh(K) smoothly slice if and only if K is slice where Wh(K) is the untwisted Whitehead double of K).

3. Rob Kirby

Problem 7. *Extend Dave Gay's beautiful proof using Morse/Cerf theory on functions on surfaces to show that 4-dimensional oriented bordism is Z, to the case of spin bordism and more generallly, characteristic bordism.*

4. Tian-Jun Li

I think the following conjecture of Gompf (in his fiber sum paper) is central:

Conjecture 1. *The Euler number of any closed symplectic 4-manifold M is non-negative, except when M is irrationally ruled over a Riemann surface of genus at least 2.*

For Kahler surfaces, this is known (a simple nice proof given in the book of BHPV). For symplectic 4-manifolds with non-positive Kodaira dim, this is also known (the - ∞ case due to Liu, the zero case due to Li and Bauer). For symplectic 4-manifolds of general type, it is a first step towards the BMY inequality.
A smaller problem is to rule out a symplectic 4-manifold with $b_2 = 1$ and $b_1 = 2$. This homology type would be the only possible counterexample in the $b_+ = 1$ case. Perhaps this problem is approachable, at least under some additional geometry assumption like symmetry. For such a manifold, the cup product on rational H^1 is clearly trivial.

5. Tye Lidman

A major problem is:

Problem 8. *To what extent does the Bogomolov-Miyaoka-Yau inequality hold for symplectic four-manifolds?*

A smaller version of this problem to start with would be:

Problem 9. *The biggest violator of the BMY inequality would be a positive-definite four-manifold with $b^+ \geq 2$. Can we prove the BMY inequality in this outrageous case?*

This is known if the manifold has a perfect Morse function, but the general case is still quite open.
This problem would be important because this would allow methods from algebraic/complex geometry to be applied to the more general setting of symplectic topology.
Another major problem is:

Problem 10. *Do smooth simply-connected closed four-manifolds admit perfect Morse functions, i.e. admit handle decompositions with no 1- or 3-handles?*

To start out:

Problem 11. *Consider a homology sphere Y with weight one fundamental group. There is a simply-connected four-manifold homotopy equivalent to S^2 obtained by attaching a 2-handle to $(Y - B^3) \times I$ along a knot representing the generator of π_1 in Y. Sometimes such a manifold can be shown to need 1-handles. Can it be shown to need 3-handles as well? This might be easier than the closed case. Also, gluing two of these manifolds together gives a four-manifold homeomorphic to S^2xS^2 or $CP^2\# - CP^2$. Do these need 1- or 3-handles?*

If everything could be reduced to two-handle attachments, it seems like this would make it more likely to be able to actually prove diffeomorphism characterizations of some four-manifolds. For example, this would immediately characterize S^4 (the Poincare conjecture) and CP^2. On the other hand, if the answer is no, this tells us that four-manifolds are even more complicated than we already know. (Although, this would not be surprising.) Perhaps the more interesting part is finding the tools/techniques to detect the necessity of such handles. This seems likely to lead to lots of new exotic four-manifolds.

6. Allison Miller

Concordance and manifolds:

Problem 12. *Many concordance obstructions come from associated 3- or 4-manifolds (cyclic branched covers, Dehn surgeries, n-traces...), so it is natural to ask to what extent these manifolds determine the concordance properties of knots. The Akbulut-Kirby conjecture (recently disproved by Yasui) falls in this category, as do questions about whether e.g. two knots whose surgeries are all pairwise homology cobordant must be concordant.*

Maps and structure on concordance:

Problem 13. *Much of the study of knot concordance has focused on its group theoretic structure, which remains poorly understood. An complementary perspective, however, comes from the additional structure of a metric or filtration, whether solvable, grope, or bipolar. Satellite induced maps are both a useful tool in understanding these structures and worthy of study in their own right- one open question is whether there are any 'interesting' satellite-induced homomorphisms on concordance.*

7. Bülent Tosun

I would like to list a few problems and a related conjecture that I have been very actively thinking about and which I find very important in the area of low dimensional contact and symplectic geometry.

Three dimensional contact geometry:

The two outstanding open problems in contact geometry in 3–dimensions concern the *existence* and *classification* of tight contact structures: Which closed, oriented 3–manifolds admit tight contact structure? When a 3–manifold admits a tight contact structure can one classify all contact structures on the manifold? A great deal of deep and beautiful work in last two decades has been put towards the resolution of these fundamental problems. But at the moment it is fair to say that a thorough understanding is far from complete. Here we focus on the existence problem, which is reduced to answering the following.

Problem 14. *Does every hyperbolic 3–manifold which is a rational homology sphere admit a tight contact structure?*

The main difficulty in answering Question 14 fully is the lack of concrete connections between hyperbolic geometry and gauge theoretical invariants, which are powerful tools to study contact and symplectic geometry.

For now, natural examples that fit well under the class of 3–manifolds in Question 14 are obtained by contact rational Dehn surgeries along a null-homologous knot in a rational homology sphere Y. Let L be an oriented Legendrian knot in $Y = S^3$ and $0 < r \in \mathbb{Q}$. Write (Y_r, ξ_r^-) for the contact manifold obtained by contact r-surgery on L, in which all stabilizations are chosen to be negative (or equivalently we consider the inadmissible transverse surgery). Let $c(\xi_r^-) \in \widehat{HF}(-Y_r)$ be the Ozsváth-Szabó contact invariant of ξ_r^-. T. Mark and I can tell exactly when $c(\xi_r^-) \neq 0$. A natural question is the following.

Problem 15. *Is it true that, $c(\xi_r^-) \neq 0$ if and only if ξ_r^- is tight?*

We note that there do exist tight contact structures with vanishing contact class, but it is unknown if any of these arises as a contact surgery on a knot.

Smooth embeddings with Stein images:

Problem 16. *Is there a Brieskorn homology sphere that embeds in $\mathbb{C}^2$ as the boundary of a Stein domain in $\mathbb{C}^2$?*

We refer to such an embedding of a 3-manifold as a *pseudoconvex embedding* in $\mathbb{C}^2$.

Question 16 is a natural extension of two classically studied problems (with tremendous impact) in low dimensional topology: Which integral homology spheres admit topological/smooth embedding in $\mathbb{R}^4$? For the topological one, there is somewhat a satisfying answer due to Freedman (1982), who proves that any integral homology sphere bounds a contractible topological 4-manifold. The answer is NO for the smooth embeddings by many people, starting with as a consequence of Rokhlin's Theorem (1952). On the other hand there are many infinite families of integral homology spheres (including Brieskorn homology spheres and 3-manifolds modeled on other geometries) that do embed in $\mathbb{R}^4$ smoothly. For example, consider smooth 1 surgery on a slice knot. It is worth mentioning though that despite many advances and lots of work done in the last 7 decades, it is still unclear, for example, which Brieskorn homology spheres embed in $\mathbb{R}^4$ smoothly, and which don't. Gompf in his "Smooth embeddings with Stein images" paper makes the following remarkable prediction for Question 16.

Conjecture 2 (Gompf). *No Brieskorn integer homology sphere (other than S^3) admits a pseudoconvex embedding in $\mathbb{C}^2$, with either orientation.*

8. Danny Ruberman

Problem 17. *Is every knot in a homology 3-sphere topologically slice in a contractible manifold (or homology ball) W?*

There is no requirement here that the slice be locally flat, although an interesting variation would require that it be locally flat except for a single cone singularity.
This is motivated by the recent interest in smooth concordance in homology cobordisms, as well as the following problem on group actions:

Problem 18. *Does every finite group action on a homology sphere extend to a group action on a contractible manifold (or homology ball) W?*

This holds for free actions; if there are fixed points on Y then there would be fixed points in W that in the cyclic case could be homeomorphic to a disk.
In relation to my talk, Id like to know the answer to

Problem 19. *Surgery theory gives rise to many topological 4-manifolds that are simple homotopy equivalent to a lens space cross with a circle, but not homeomorphic to any manifold of that form. Are such manifolds smoothable?*

Problem 20. *Develop obstructions to embedding punctured 3-manifolds (i.e. $Y - B^3$ where Y is closed) in the 4-sphere (topologically, or especially smoothly). This is of particular interest when Y is not a rational homology sphere. For example, if Y is 0-surgery on a knot K, then (Kawauchi 1978) a punctured embedding places restrictions on the signature function of K. What does it imply about other concordance invariants of K, eg does K have to be slice in order to have such an embedding?*