Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Large-deviation principles in SINR-based wireless network models

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joint work with Benedikt Jahnel, Paul Keeler and Robert Patterson

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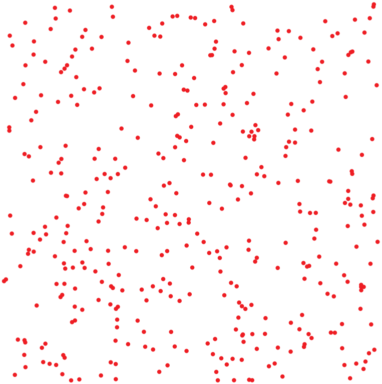
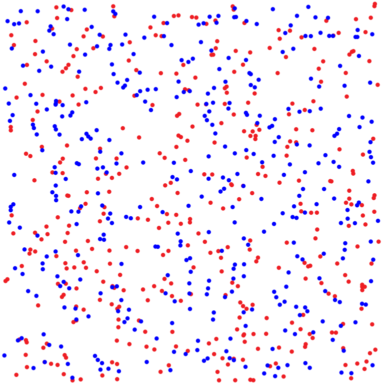
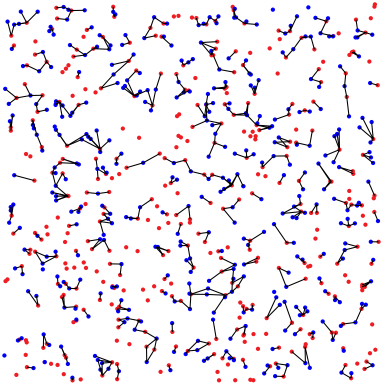


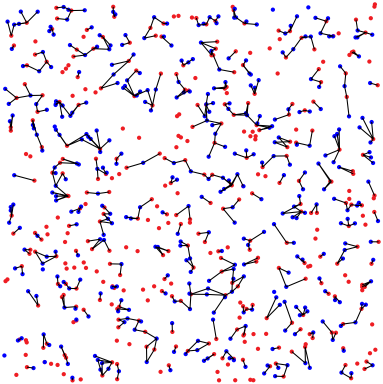
- homogeneous Poisson point process of *transmitters*



- homogeneous Poisson point process of *transmitters*
- homogeneous Poisson point process of *receivers*



- homogeneous Poisson point process of *transmitters*
- homogeneous Poisson point process of *receivers*
- *connectable receivers* associated with each *transmitter*



- homogeneous Poisson point process of *transmitters*
- homogeneous Poisson point process of *receivers*
- *connectable receivers* associated with each *transmitter*
- *goal. exponential decay* of probability of *unlikely configurations*
  - e.g., large proportion of *isolated transmitters*

- *network participants*
  - $X =$ homogeneous Poisson point process of *transmitters* in $\mathbb{R}^d$; intensity λ_{T}
  - $Y =$ homogeneous Poisson point process of *receivers* in $\mathbb{R}^d$; intensity λ_{R}

- *network participants*
  - $X =$ homogeneous Poisson point process of *transmitters* in $\mathbb{R}^d$; intensity λ_T
  - $Y =$ homogeneous Poisson point process of *receivers* in $\mathbb{R}^d$; intensity λ_R
- *SINR model*

$$\mathsf{SINR}(X_i, Y_j) = \mathsf{SINR}(X_i, Y_j, X) = \frac{P_{X_i} F_{X_i,Y_j} \ell(|X_i - Y_j|)}{w + \sum_{k \neq i} P_{X_k} F_{X_k,Y_j} \ell(|X_k - Y_j|)}$$

  - $w =$ constant *thermal noise*
  - $\ell(r) = \min\{1, r^{-\alpha}\}$ *path-loss function* with *path-loss exponent* $\alpha > d$
  - $\{P_x\}_{x \in X} =$ iid *transmission powers*
    - compact support
  - $\{F_{x,y}\}_{x \in X,\, y \in Y} =$ iid *fading variables*
    - compact support
    - cdf q of $F_{x,y}^{-1}$ globally Lipschitz

- *connectable receivers* $Y^{(i)}$ associated with transmitter X_i

$$Y^{(i)} = \{Y_j \in Y : \mathsf{SINR}(X_i, Y_j) \geq \tau\}$$

- possible applications
  - D2D networks
  - first step in a multi-hop network

Reachable receivers

- *connectable receivers* $Y^{(i)}$ associated with transmitter X_i

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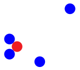
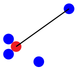
- possible applications
  - D2D networks
  - first step in a multi-hop network
- *goal. exponential decay* of probability that untypically large proportion of transmitters suffers from some *frustration event*



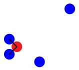
$Y^{(i)} = \emptyset$

isolation



$\#Y^{(i)} \leq 1$

at most 1 *connectable receiver*



$Y^{(i)} \subset B_\varepsilon(X_i)$

impossibility of long hops

- *empirical measure of connectable receivers*

$$L_n = \frac{1}{|\Lambda_n|} \sum_{X_i \in \Lambda_n} \delta_{Y^{(i)} - X_i}, \qquad \Lambda_n = [-n/2, n/2]^d$$

- e.g. $L_n(\{\emptyset\}) = \frac{1}{|\Lambda_n|} \#\{X_i \in \Lambda_n : X_i$ *isolated*$\}$

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- e.g. $L_n(\{\emptyset\}) = \frac{1}{|\Lambda_n|} \#\{X_i \in \Lambda_n : X_i$ *isolated*$\}$
- for $\mathbb{Q} \in \mathcal{P}_\theta$ any *jointly* stationary point process of transmitters and receivers, let

$$\mathbb{Q}^*(\cdot) = \mathbb{E}\#\{X_i \in \Lambda_1 : Y^{(i)} - X_i \in \cdot\}$$

  be the *Palm mark measure* of $\{(X_i, Y^{(i)} - X_i)\}_{i \geq 1}$
- specific relative entropy / Kullback-Leibler distance

$$h(\mathbb{Q}|\mathbb{P}) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \mathbb{E}_{\mathbb{P}}\Big(\frac{\mathrm{d}\mathbb{Q}|_{\Lambda_n}}{\mathrm{d}\mathbb{P}|_{\Lambda_n}} \log \frac{\mathrm{d}\mathbb{Q}|_{\Lambda_n}}{\mathrm{d}\mathbb{P}|_{\Lambda_n}}\Big).$$

Theorem

The random measures $\{L_n\}_{n\geq 1}$ satisfy an LDP in the τ-topology with speed $|\Lambda_n|$ and good rate function

$$\mathcal{I}(Q) = \inf_{\substack{\mathbb{Q}\in\mathcal{P}_\theta \\ \mathbb{Q}^*=Q}} h(\mathbb{Q}|\mathbb{P}).$$

Loosely speaking,

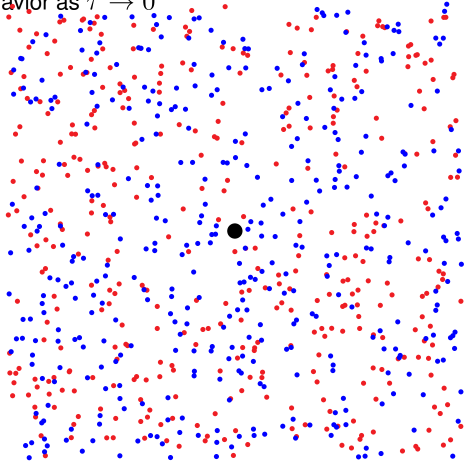
$$\mathbb{P}(L_n \in F) \approx \textit{exp}(-|\Lambda_n| \inf_{Q\in F} \mathcal{I}(Q)).$$

- τ-topology generated by $B \mapsto \mu(B)$, B Borel set
- LDPs/importance sampling for interference in (Ganesh & Torrisi, 2008) or (Torrisi & Leonardi, 2013)
- proof based on lvl-3 LDP for Poisson pt. processes (Georgii & Zessin, 1993)

- Y^τ = connectable receivers for distinguished vertex at o

$$Y^\tau = \{Y_j \in Y : \mathrm{SINR}(o, Y_j) \geq \tau\}$$

- asymptotic behavior as $\tau \to 0$



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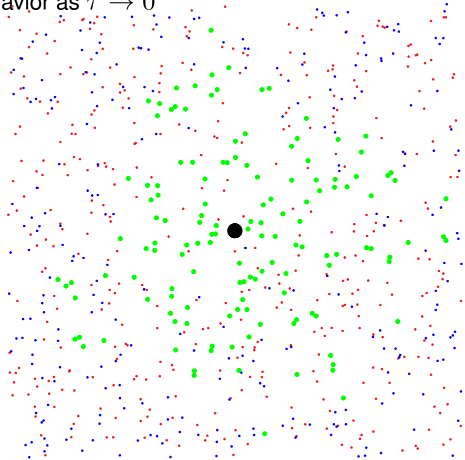

Theorem

As $\tau \to 0$ the random measure $\{\tau^{\frac{d}{\alpha}} Y^\tau(\tau^{-\frac{1}{\alpha}} B)\}_{B \in \mathcal{B}(\Lambda_1)}$ satisfies an LDP in the weak topology with rate $\tau^{-\frac{d}{\alpha}}$ and good rate function given by

$$\mathcal{I}(\mu) = \begin{cases} \int_{\Lambda_1} \mathcal{I}_y(f(y))\mathrm{d}y & \text{if } f = \mathrm{d}\mu/\mathrm{d}x \text{ exists,} \\ \infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{I}_y(s) = \inf_{\mathbb{Q} \in \mathcal{P}_\theta} \Big(h(\mathbb{Q}|\mathbb{P}) + h\big(\mathsf{Pois}(s)|\mathsf{Pois}(\lambda_\mathsf{R} \mathbb{E}_\mathbb{Q}[\mathbb{P}(\mathsf{SINR}(o,y) \geq 1|X)])\big)\Big).$$

- *idea.* Y^τ is *Cox point process* with random intensity measure $B \mapsto \lambda_\mathsf{R} \int_B \mathbb{P}(\mathsf{SINR}(o,y) \geq \tau|X)\mathrm{d}y$ + (Georgii & Zessin, 1993) + (Dawson & Gärtner)

Corollary

$$\lim_{\tau\to 0}\tau^{\frac{d}{\alpha}}\log\mathbb{P}(Y^\tau=\emptyset)=\lim_{\tau\to 0}\tau^{\frac{d}{\alpha}}\log\mathbb{E}\textit{exp}\Big(-\lambda_{\mathsf{R}}\int_{\Lambda_{\tau^{-1/\alpha}}}\mathbb{P}(\mathsf{SINR}(o,y)\ge\tau|X)\mathrm{d}y\Big)$$

$$=-\int_{\Lambda_1}\inf_{\mathbb{Q}\in\mathcal{P}_\theta}\big(h(\mathbb{Q}|\mathbb{P})+\lambda_{\mathsf{R}}\mathbb{E}_{\mathbb{Q}}[\mathbb{P}(\mathsf{SINR}(o,y)\ge 1|X)]\big)\mathrm{d}y.$$

- variational characterization of rate function *intractable* analytically and numerically
- minimization problem can be solved for class of Poisson point processes

$\Rightarrow$ importance-sampling algorithms

- *problem.* estimate isolation probability p_τ for small τ by Monte Carlo method
- accurate estimation of p_τ by naïve MC requires *large number of simulation runs*
- *idea.* perform simulation under *different probability measure* $\mathbb{Q}$ that makes rare event *more likely*

$$\mathbb{P}(o \text{ is isolated}) = \mathbb{Q}\Big(\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(\omega)\mathbb{1}_{o \text{ is isolated}}(\omega)\Big).$$

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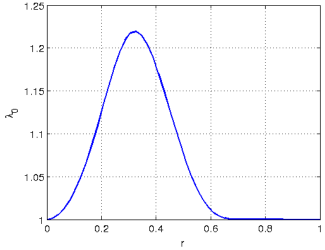
- *heuristic.* optimal $\mathbb{Q} \approx$ conditional distribution under rare event
- *connection to LDPs.* use minimizers in rate function
- $\alpha = 4$, $w = 1$, constant fading, constant transmission powers

$$\lim_{\tau\to 0}\tau^{1/2}\log p_\tau = -\int_{\Lambda_1}\inf_{\mathbb{Q}\in\mathcal{P}_\theta}\big(h(\mathbb{Q}|\mathbb{P}) + \lambda_{\mathsf{R}}\mathbb{Q}(\mathsf{SINR}(o,y)\geq 1)\big)\mathrm{d}y.$$

- optimization only over *inhomogeneous Poisson point processes*

$\Rightarrow$ continuous family of 1D-optimization problems for optimal intensity $\lambda_{\mathsf{opt}}(r)$



- $N = 1,000,000$ simulation runs
- variance reduction by 78%

	isolation probability	variance
$\lambda(\cdot) \equiv 1$	$7.72 \cdot 10^{-6}$	$8.22 \cdot 10^{-10}$
$\lambda(\cdot) \equiv \lambda_{\mathsf{opt}}(\cdot)$	$7.70 \cdot 10^{-6}$	$1.78 \cdot 10^{-10}$

Thank you for your attention!

[1] F. Baccelli and B. Błaszczyszyn.
Stochastic Geometry and Wireless Networks: Volume 1: Theory.
Now Publishers Inc, 2009.

[2] A. J. Ganesh and G. L. Torrisi.
Large deviations of the interference in a wireless communication model.
IEEE Trans. Inform. Theory, 54(8):3505–3517, 2008.

[3] H.-O. Georgii and H. Zessin.
Large deviations and the maximum entropy principle for marked point random fields.
Probab. Theory Related Fields, 96(2):177–204, 1993.

[4] G. L. Torrisi and E. Leonardi.
Simulating the tail of the interference in a Poisson network model.
IEEE Trans. Inform. Theory, 59(3):1773–1787, 2013.