Asymptotics and Meta-Distribution of the Signal-to-Interference Ratio in Wireless Networks
Part II

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Part II

Overview

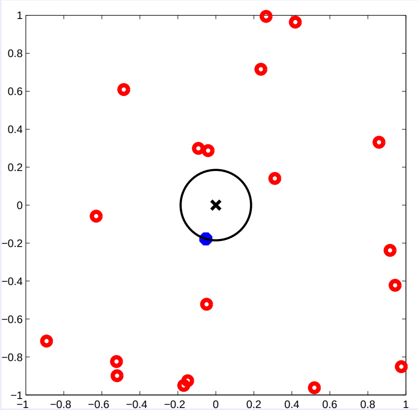
- Cellular networks and the HIP model
- Standard analysis of some transmission techniques for the PPP
- Non-Poisson network analysis using ASAPPP[a]
  - The idea of the horizontal shift (gain) of SIR distributions
  - The relative distance process
  - The MISR[b] and the EFIR[c]
  - Asymptotic gains at 0 and ∞
  - Examples
- Concluding remarks
- Homework?

[a] Approximate SIR Analysis based on the PPP—or simply "as a PPP"
[b] Mean interference-to-signal ratio
[c] Expected fading-to-interference ratio

From bipolar to cellular networks

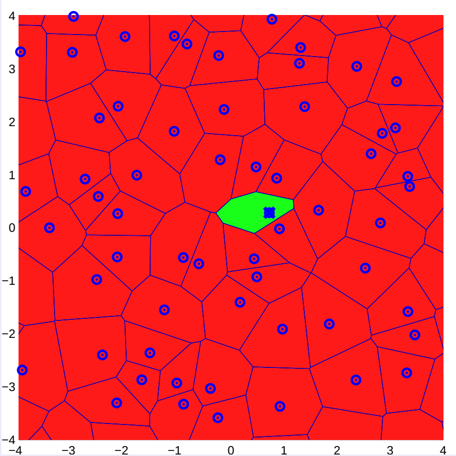
From yesterday: A generic cellular network (downlink)

- Base stations form a stationary and ergodic point process and all transmit at equal power.
- Assume a user is located at o. Its serving base station is the nearest one (strongest on average).
- The other base stations are interferers (frequency reuse 1).



Single-tier cellular networks with reuse 1

SIR with strongest-base station (BS) association



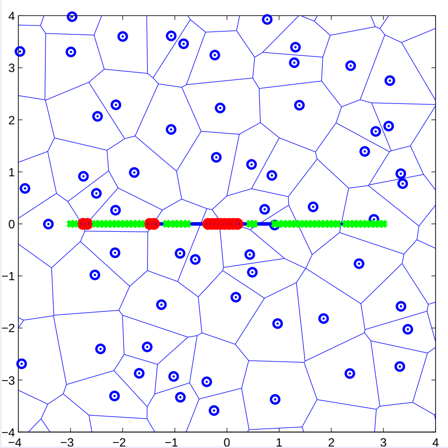
$$\mathrm{SIR} \triangleq \frac{\mathsf{S}}{\mathsf{I}}$$

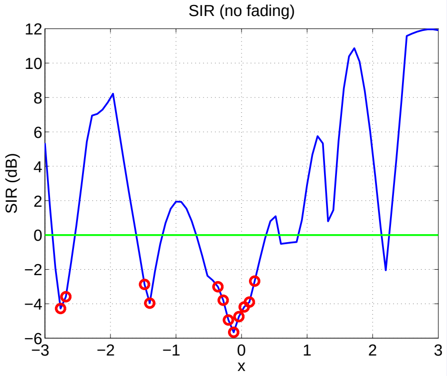
$$\mathsf{S} = h\|x_0\|^{-\alpha}$$

$$\mathsf{I} = \sum_{x\in\Phi\setminus\{x_0\}} h_x\|x\|^{-\alpha}$$

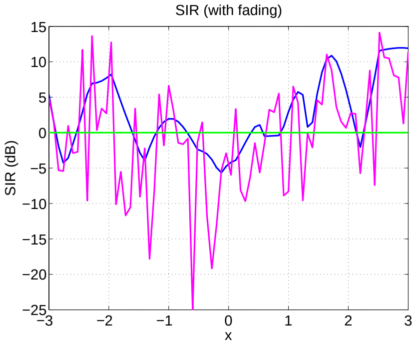
Φ: point process of BSs
x_0: serving BS
h, (h_x): iid fading

The SIR walk and coverage at 0 dB





SIR distribution



The fraction of a long curve (or large region) that is above the threshold θ is the ccdf of the SIR at θ:

$$p_s(\theta) \triangleq \bar{F}_{\mathrm{SIR}}(\theta) \triangleq \mathbb{P}(\mathrm{SIR} > \theta)$$

It is the fraction of the users with SIR $> \theta$ for each realization of the BS and user processes.

Fact on SIR distributions

Only the PPP is tractable exactly—in some cases

If the base stations form a homogeneous Poisson point process (PPP):

$$p_{\mathrm{s}}(\theta) \triangleq \bar{F}_{\mathrm{SIR}}(\theta) = \frac{1}{{}_2F_1(1,-\delta;1-\delta;-\theta)}, \quad \delta \triangleq 2/\alpha.$$

For $\delta = 1/2$, $p_{\mathrm{s}}(\theta) = \left(1+\sqrt{\theta}\arctan\sqrt{\theta}\right)^{-1}$.

If the fading is not Rayleigh or if the point process is not Poisson, it gets hard very quickly.

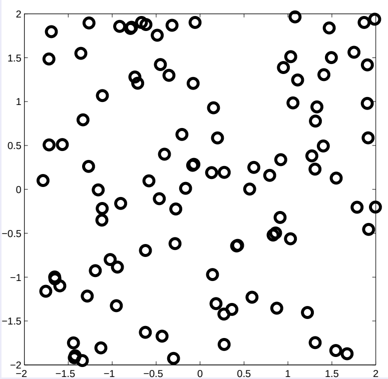
So let us enjoy the beauty of Poissonia a little longer.

Poissonia

The HIP baseline model for HetNets

The HIP (homogeneous independent Poisson) model[a]

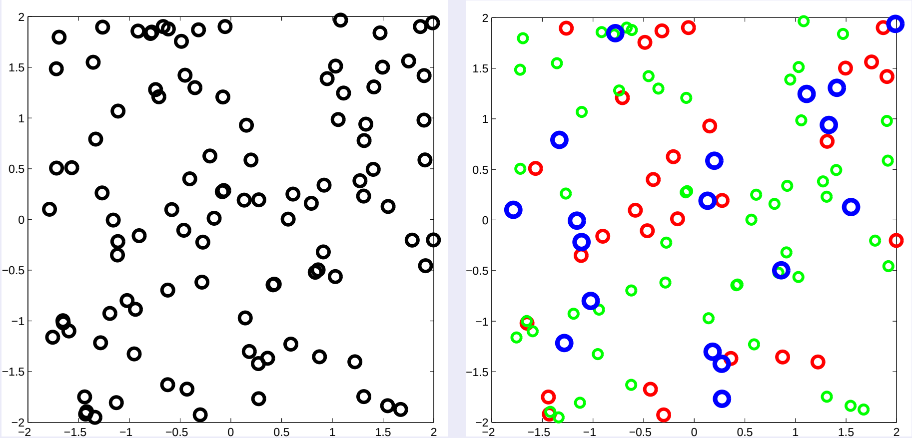
[a] Dhillon et al., "Modeling and Analysis of K-Tier Downlink Heterogeneous Cellular Networks". 2012.



- Start with a homogeneous PPP. Here $\lambda = 6$.
- Choose a number of tiers and intensities for each tier, say $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$.
- Then randomly color the BSs according to the intensities to assign them to the different tiers:

$$\mathbb{P}(\text{tier} = i) = \lambda_i / \lambda$$

The HIP (homogeneous independent Poisson) model



Here $\lambda_i = 1, 2, 3$. Assign power levels P_i to each tier.

This model is doubly independent and thus highly tractable.

Equivalence of all HIP models

From the perspective of the typical user, this network is completely equivalent to a single-tier Poisson model with unit power and unit density.

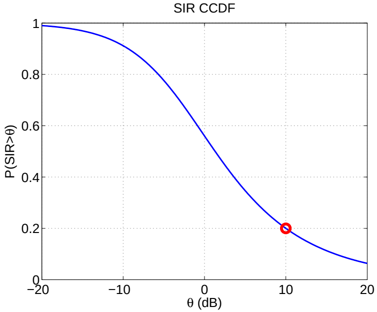
Hence for all HIP models (with Rayleigh fading and power law path loss), the SIR distribution is

$$p_s(\theta) \triangleq \bar{F}_{\mathrm{SIR}}(\theta) = \frac{1}{{}_2F_1(1,-\delta;1-\delta;-\theta)}.$$

In particular, for $\delta = 1/2$:

$$p_s(10) = 20.00\%$$

The typical user is not impressed with this performance.



$\alpha = 4$ $(\delta = 1/2)$.

Explanation for equivalence

For a single tier with unit transmit power, let

$$\Xi = \{\xi_i\} \triangleq \{x \in \Phi \colon \|x\|^\alpha / h_x\}.$$

The received powers from the nodes in Φ are $\{\xi^{-1}\}$.
If $\Phi \subset \mathbb{R}^2$ is Poisson with intensity λ, then Ξ is Poisson with intensity function $\mu(r) = \lambda\pi\delta r^{\delta-1}\mathbb{E}(h^\delta)$.

For multiple independent Poisson tiers with transmit power P_k, the union

$$\Xi = \{\xi_i\} = \bigcup_{k\in[K]} \{x \in \Phi_k \colon \|x\|^\alpha / (P_k h_x)\}.$$

is a PPP with intensity function

$$\mu(r) = \sum_{k\in[K]} \pi\lambda_k \delta P_k^\delta r^{\delta-1}\mathbb{E}(h^\delta)\,.$$

In any case, $\mu(r) \propto r^{\delta-1}$. The pre-constant does not matter for the SIR.

Path loss process

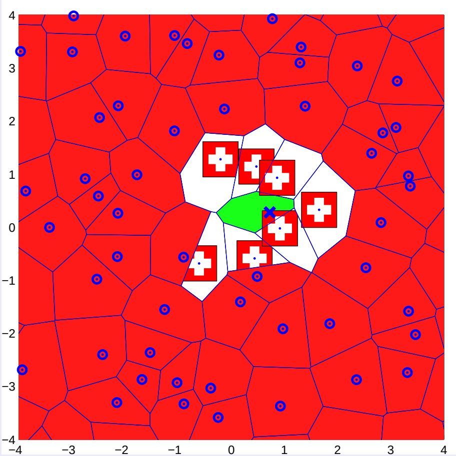
- The point process $\Xi = \{\xi_i\} \subset \mathbb{R}^+$ where $\{\xi_i^{-1}\}$ are the received powers (with or without fading) is called the path loss process or propagation process.
- It is a key ingredient in many proofs of many results for cellular networks[a] and HetNets[b].
- The equivalence also holds for advanced transmission techniques, such as BS cooperation and silencing.

Let us have a look at some of these advanced techniques.

[a] Blaszczyszyn, Karray, and Keeler, "Using Poisson Processes to Model Lattice Cellular Networks". 2013.

[b] Zhang and Haenggi, "A Stochastic Geometry Analysis of Inter-cell Interference Coordination and Intra-cell Diversity". 2014; Nigam, Minero, and Haenggi, "Coordinated Multipoint Joint Transmission in Heterogeneous Networks". 2014.

BS silencing: neutralize nearby foes



The strongest BS (on average) is the serving BS, while the $n-1$ next-strongest ones are silenced. The model may include shadowing (which stays constant over time).

SIR distribution for silencing (ICIC)

With BS silencing (or inter-cell interference coordination, ICIC) of $n-1$ BSs, the SIR distribution is[a]

$$p_s^{(!n)} \triangleq \mathbb{P}\left(\frac{\text{power from serving BS}}{\text{power from BSs } \textit{beyond} \text{ the } n^{\text{th}}} > \theta\right) = (n-1)\delta \int_0^1 \frac{(1-x^\delta)^{n-2} x^{\delta-1}}{(C_1(\theta x,1))^n} \mathrm{d}x\,,$$

where $C_1(s,m) = {}_2F_1(m,-\delta;1-\delta;-s)$.

This result does not depend on the shadowing distribution—as long as its δ-th moment is finite.

[a] Zhang and Haenggi, "A Stochastic Geometry Analysis of Inter-cell Interference Coordination and Intra-cell Diversity". 2014.

Intra-cell diversity from multiple resource blocks

Transmission over M resource blocks

Here, all base stations interfere, but the serving one uses M resource blocks (with independent fading) to serve the user.

The success probability is the probability that the SIR in at least one of them exceeds θ:

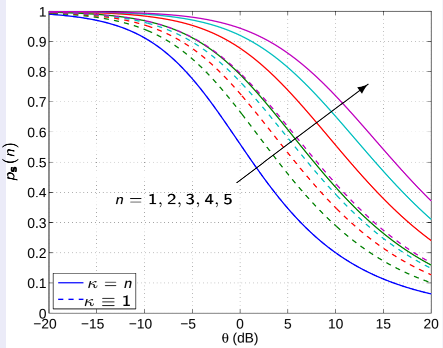
$$p_s^{(\cup M)} \triangleq \mathbb{P}\left(\bigcup_{m=1}^{M} S_m\right), \quad \text{where } S_m = \{\mathrm{SIR}_m > \theta\}.$$

For the joint success probability, we have

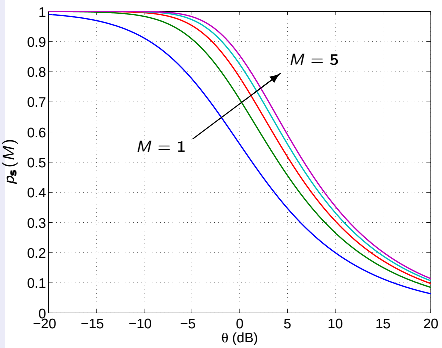
$$p_s^{(\cap M)} \triangleq \mathbb{P}\left(\bigcap_{m=1}^{M} S_m\right) = \frac{1}{C_1(\theta, M)} = \frac{1}{{}_2F_1(M, -\delta; 1-\delta; -\theta)}.$$

$p_s^{(\cup M)}$ follows from inclusion/exclusion.

n-BS silencing (ICIC) vs. transmission over M RBs (ICD)



n-BS ICIC. $\alpha = 4$.



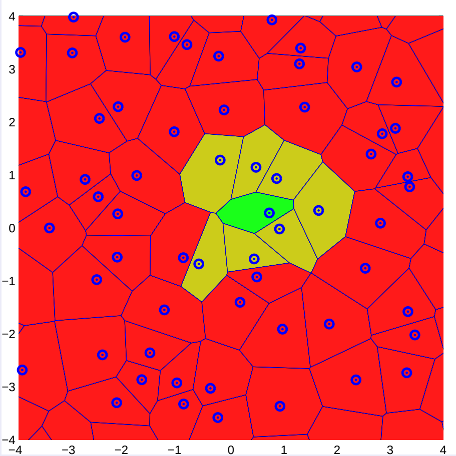
M-RB ICD. $\alpha = 4$.

Observation

ICIC provides a gain in the SIR but no diversity. ICD has a diversity gain of M. As a result, ICD is superior at small values of θ ($\theta < -5$ dB).

Cooperation by joint transmission

BS cooperation: turn nearby foes into friends



SIR distribution with BS cooperation

- In the HIP model, let the users receive combined signals from the n strongest (on average) BSs, denoted by $\mathcal{C}$.
- Channels are Rayleigh fading, and BSs use non-coherent joint transmission.
- The amplitude fading coefficients (g_x) are zero-mean unit-variance complex Gaussian, and the signal power is

$$S = \Big| \sum_{x \in \mathcal{C}} g_x \sqrt{P_x} \|x\|^{-\alpha/2} \Big|^2 .$$

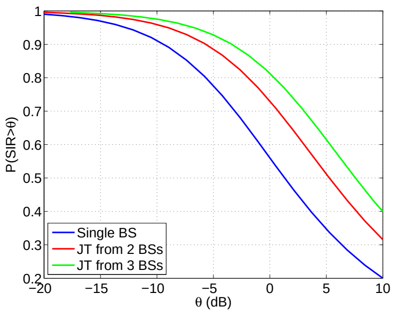
  S is exponentially distributed with mean $\sum P_x \|x\|^{-\alpha}$.
- The interference stems from $\Phi \setminus \mathcal{C}$.

BS cooperation with non-coherent JT

Let

$$\mathbf{u} = (u_1, \dots, u_n)$$
$$\tilde{\mathbf{u}} = (u_n/u_1, \dots, u_n/u_n)$$
$$Z(\mathbf{u}) = \|\tilde{\mathbf{u}}\|_{\alpha/2}\theta^{-\delta}$$
$$F(x) = \int_x^\infty \frac{r}{1+r^\alpha}\mathrm{d}r$$

$\alpha = 4$



The success probability is independent of power levels and densities[a]

$$p_s(\theta) = \int_{0<u_1<\dots<u_n<\infty} \exp\left(-u_n\left(1+2\frac{F(\sqrt{Z(\mathbf{u})})}{Z(\mathbf{u})}\right)\right)\mathrm{d}\mathbf{u}.$$

[a] Nigam, Minero, and Haenggi, "Coordinated Multipoint Joint Transmission in Heterogeneous Networks". 2014.

What is possible outside Poissonia?

Ginibre point process (GPP)

For GPP with Rayleigh fading[a]: $p_{\mathrm{s}}(\theta) =$

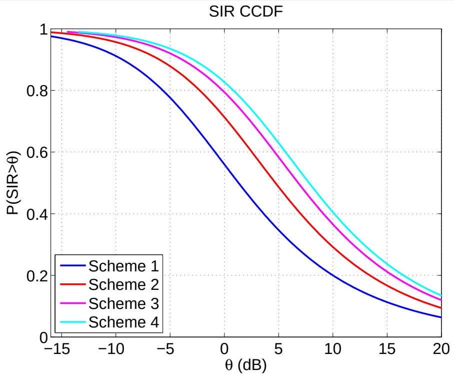
$$\int_0^\infty e^{-v}\left[\prod_{j=0}^{\infty}\frac{1}{j!}\int_v^\infty \frac{s^j e^{-s}}{1+\theta(v/s)^{\alpha/2}}\mathrm{d}s\right]\left[\sum_{i=0}^{\infty} v^i \left(\int_v^\infty \frac{s^i e^{-s}}{1+\theta(v/s)^{\alpha/2}}\mathrm{d}s\right)^{-1}\right]\mathrm{d}v$$

[a] Miyoshi and Shirai, "A Cellular Network Model with Ginibre Configured Base Stations". 2014.

Observation on SIR distributions

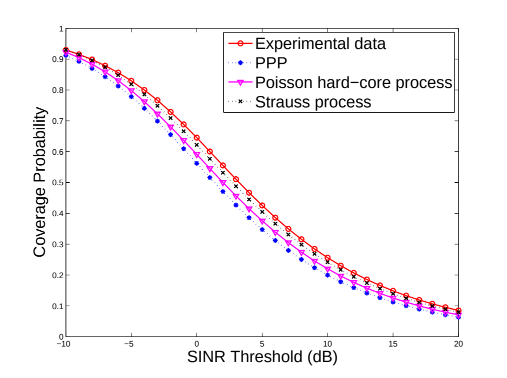
Shape of SIR distributions

In many cellular papers, we find figures like this:



It appears that: The curves all have the same shape—they are merely shifted horizontally!

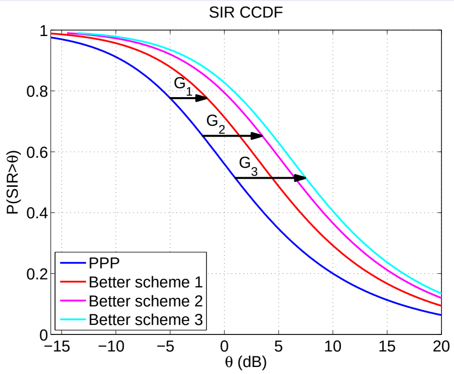
Different BS point processes



Indeed—visually, the curves are shifts of each other. Since the shift (or gain) is due to the deployment, we call it deployment gain[a].

[a] Guo and Haenggi, "Asymptotic Deployment Gain: A Simple Approach to Characterize the SINR Distribution in General Cellular Networks". 2015.

ASAPPP: Approximate SIR analysis based on the PPP



If the SIR ccdfs were indeed just shifted:

$$p_{\mathrm{s,PPP}}(\theta) \triangleq \mathbb{P}(\mathrm{SIR}_{\mathrm{PPP}} > \theta) \quad \Rightarrow \quad p_{\mathrm{s}}(\theta) = p_{\mathrm{s,PPP}}(\theta/G).$$

G is the SIR shift (in dB) or the SIR gain or gap.

Horizontal gap and asymptotics

The shift at threshold θ is

$$G(\theta) \triangleq \frac{\bar{F}_{\mathrm{SIR}}^{-1}(p_{\mathrm{s,PPP}}(\theta))}{\theta},$$

hence we have $p_{\mathrm{s}}(\theta) = p_{\mathrm{s,PPP}}(\theta/G(\theta))$.

The asymptotic gains are

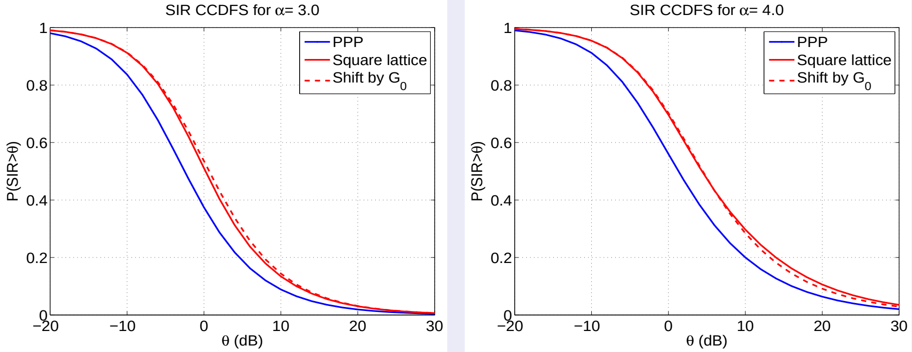
$$G_0 \triangleq \lim_{\theta\downarrow 0} G(\theta); \quad G_\infty \triangleq \lim_{\theta\uparrow\infty} G(\theta).$$

So (if G_0 and G_∞ exist),

$$p_{\mathrm{s}}(\theta) \sim p_{\mathrm{s,PPP}}(\theta/G_0), \quad \theta \to 0\,; \qquad p_{\mathrm{s}}(\theta) \sim p_{\mathrm{s,PPP}}(\theta/G_\infty), \quad \theta \to \infty.$$

Observation: $G(\theta) \approx G_0$ for all θ, i.e., a shift by G_0 results in an approximation that is quite accurate over the entire distribution.

Example 1: Deployment gain of square lattice



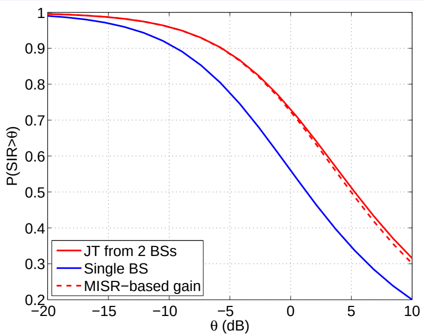
For the square lattice:

$$G_0 = 3.19 \text{ dB for } \alpha = 3 \text{ and } G_0 = 3.14 \text{ dB for } \alpha = 4.$$

So applying a gain of 2 yields an accurate approximation. For $\alpha = 4$,

$$p_s^{sq}(\theta) \approx \left(1 + \sqrt{\theta/2} \arctan \sqrt{\theta/2}\right)^{-1}.$$

Example 2: Gain of joint transmission



Again the ccdf for the cases without and with cooperation are very similar in shape.

The shift here is $G_0 = 2/(4 - \pi) \approx 2.33$.

The ISR and the MISR

Definition ($\mathrm{I\bar{S}R}$)

The interference-to-average-signal ratio is

$$\mathrm{I\bar{S}R} \triangleq \frac{I}{\mathbb{E}_h(S)},$$

where $\mathbb{E}_h(S)$ is the desired signal power averaged over the fading.

Remarks

- The $\mathrm{I\bar{S}R}$ is a random variable due to the random positions of BSs and users. Its mean MISR is a function of the network geometry only.
- If the interferers are located at distances R_k,

$$\mathrm{MISR} \triangleq \mathbb{E}(\mathrm{I\bar{S}R}) = \mathbb{E}\left(R^\alpha \sum h_k R_k^{-\alpha}\right) = \sum \mathbb{E}\left(\frac{R}{R_k}\right)^\alpha .$$

Relevance of the MISR for Rayleigh fading

$$p_{\mathrm{out}}(\theta) = \mathbb{P}(hR^{-\alpha} < \theta I) = \mathbb{P}(h < \theta\,\mathrm{I\bar{S}R})$$

Since h is exponential, letting $\theta \to 0$,

$$\mathbb{P}(h < \theta\,\mathrm{I\bar{S}R} \mid \mathrm{I\bar{S}R}) \sim \theta\,\mathrm{I\bar{S}R} \quad \Rightarrow \quad \mathbb{P}(h < \theta\,\mathrm{I\bar{S}R}) \sim \theta\,\mathrm{MISR}.$$

So the asymptotic gain at 0 is the ratio of the two MISRs[a]:

$$G_0 = \frac{\mathrm{MISR}_{\mathrm{PPP}}}{\mathrm{MISR}}$$

The MISR for the PPP is easily calculated to be

$$\mathrm{MISR}_{\mathrm{PPP}} = \frac{2}{\alpha - 2} = \frac{\delta}{1-\delta} = \delta + \delta^2 + \delta^3 + \dots\,.$$

[a] Haenggi, "The Mean Interference-to-Signal Ratio and its Key Role in Cellular and Amorphous Networks". 2014.

ASAPPP

The method of approximating the SIR ccdf by shifting the PPP's ccdf is called ASAPPP—"Approximate SIR analysis based on the PPP".

Can we explain the unreasonable effectiveness of ASAPPP?

- Can we calculate G_0 and G_∞? How close are they?
- Can we show that the shape of the SIR distributions are similar by comparing the asymptotics?
- How sensitive are the gains to the path loss exponent and the fading model?

Some of these question are addressed in (very) recent work with Radha K. Ganti[a].

[a] Ganti and Haenggi, "Asymptotics and Approximation of the SIR Distribution in General Cellular Networks". 2015, arXiv.

RDP and MISR

Definition (The relative distance process (RDP))

For a stationary point process Φ with $x_0 = \arg\min\{x \in \Phi \colon \|x\|\}$, let

$$\mathcal{R} \triangleq \{x \in \Phi \setminus \{x_0\} \colon \|x_0\|/\|x\|\} \subset (0,1).$$

MISR using the RDP

We have

$$\mathrm{I\bar{S}R} = \sum_{y \in \mathcal{R}} h_y y^\alpha$$

and

$$\mathrm{MISR} = \mathbb{E} \sum_{y \in \mathcal{R}} y^\alpha = \int_0^1 r^\alpha \Lambda(\mathrm{d}r).$$

For the stationary PPP, $\Lambda(\mathrm{d}r) = 2r^{-3}\mathrm{d}r$.

Pgfl and moment densities of the RDP of the PPP

For the PPP, the probability generating functional (pgfl) of the RDP is

$$G_{\mathcal{R}}[f] \triangleq \mathbb{E} \prod_{x\in\mathcal{R}} f(x) = \frac{1}{1+2\int_0^1 (1-f(x))x^{-3}\mathrm{d}x},$$

and the moment densities are

$$\rho^{(n)}(t_1, t_2, \ldots, t_n) = n!\, 2^n \prod_{i=1}^{n} t_i^{-3}.$$

Pgfl for general BS processes

For a general stationary process Φ, the pgfl can be expressed as

$$G_{\mathcal{R}}[f] = \lambda \int_{\mathbb{R}^2} \mathcal{G}_o^! \left[f\left(\frac{\|x\|}{\|\cdot+x\|}\right) \mathbf{1}(\cdot + x \in b(o, \|x\|)^{\mathrm{c}}) \right] \mathrm{d}x.$$

Generalized MISR

We define

$$\mathrm{MISR}_n \triangleq (\mathbb{E}(\mathrm{I\bar{S}R}^n))^{1/n}.$$

For a Poisson cellular network with arbitrary fading,

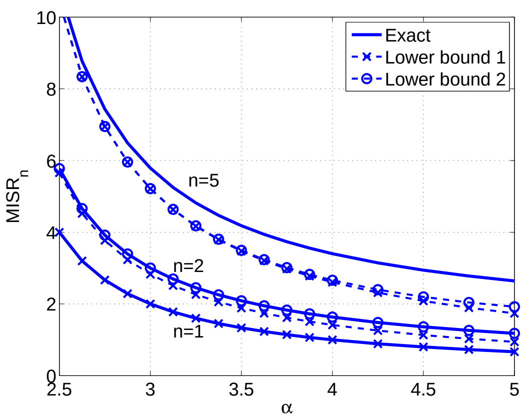
$$\mathbb{E}(\mathrm{I\bar{S}R}^n) = \sum_{k=1}^{n} k!B_{n,k}\left(\frac{\delta}{1-\delta},\ldots,\frac{\delta\mathbb{E}(h^{n-k+1})}{n-k+1-\delta}\right),$$

where $B_{n,k}$ are the Bell polynomials. A good lower bound on MISR_n is obtained by only considering the term $n = k$ in the sum:

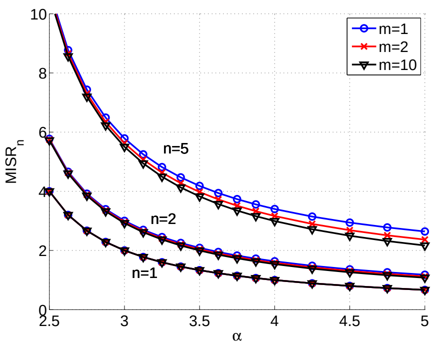
$$\mathrm{MISR}_n \geq \mathrm{MISR}_1(n!)^{1/n} = \frac{\delta}{1-\delta}(n!)^{1/n}$$

The bound does not depend on the fading. For $\delta \to 1$ ($\alpha \to 2$), it is asymptotically tight.

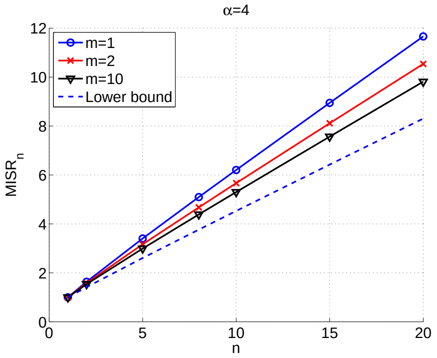
Generalized MISR for PPP with Rayleigh fading



Generalized MISR for PPP with Nakagami-m fading



MISR_n as a function of α



MISR_n as a function of n for $\alpha = 4$

For the PPP, MISR_n is essentially proportional to n. For Rayleigh fading, $\mathrm{MISR}_n \sim (n/e)\mathrm{MISR}_1$, $n \to \infty$.

Gain G_0 for general fading

If $F_h(x) \sim c_m x^m$, $x \to 0$,

$$p_s(\theta) \sim 1 - c_m \, \mathbb{E}[(\theta \, \bar{\mathrm{ISR}})^m], \quad \theta \to 0,$$

and thus

$$G_0^{(m)} = \left(\frac{\mathbb{E}(\bar{\mathrm{ISR}}^m_{\mathrm{PPP}})}{\mathbb{E}(\bar{\mathrm{ISR}}^m)} \right)^{1/m} = \frac{\mathrm{MISR}_{m,\mathrm{PPP}}}{\mathrm{MISR}_m}.$$

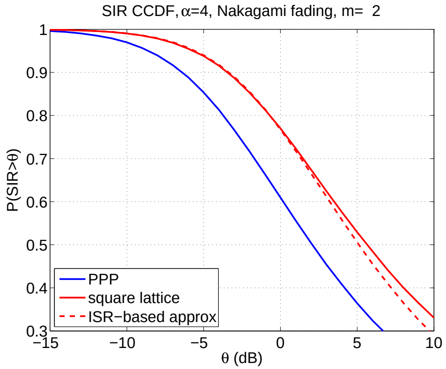
The ASAPPP approximation follows as

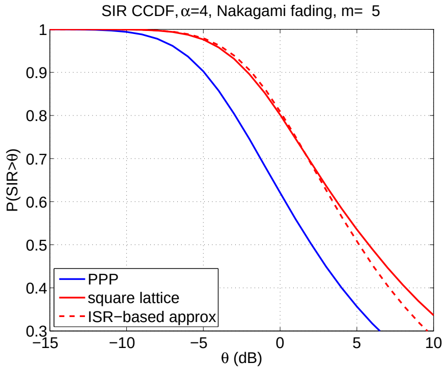
$$p_s^{(m)}(\theta) \approx p_{s,\mathrm{PPP}}^{(m)}(\theta / G_0^{(m)}).$$

This applies more generally to any transmission scheme with diversity m.

If MISR_m grows roughly in proportion to MISR_1, $G_0^{(m)} \approx G_0$, and G_0 is insensitive to the fading statistics.

Deployment gain for lattice with Nakagami fading





Here the gain for $m = 1$ (Rayleigh fading) is applied, which is 3 dB. Indeed $G_0^{(m)} \approx G_0$ in this case.

How about G_∞? Is it close to G_0?

EFIR

Definition (Expected fading-to-interference ratio (EFIR))

Let $I_\infty \triangleq \sum_{x\in\Phi} h_x\|x\|^{-\alpha}$ and let h be a fading random variable independent of all (h_x). The *expected fading-to-interference ratio* (EFIR) is defined as

$$\mathrm{EFIR} \triangleq \left(\lambda\pi\mathbb{E}_o^!\left[\left(\frac{h}{I_\infty}\right)^\delta\right]\right)^{1/\delta}, \quad \delta \triangleq 2/\alpha,$$

where $\mathbb{E}^{!o}$ is the expectation w.r.t. the reduced Palm measure of Φ.

EFIR properties

The EFIR does not depend on λ, since $\mathbb{E}_o^!(I_\infty^{-\delta}) \propto 1/\lambda$. It does not depend on the distribution of the distance to the serving BS, either.

For the PPP with arbitrary fading:

$$\mathrm{EFIR}_{\mathrm{PPP}} = (\operatorname{sinc}\delta)^{1/\delta} = (\operatorname{sinc}(2/\alpha))^{\alpha/2} \lessapprox 1-\delta.$$

SIR tail and G_∞

Theorem (SIR tail)

For all stationary BS point processes Φ, where the typical user is served by the nearest BS, with arbitrary fading,

$$p_s(\theta) \sim \left(\frac{\theta}{\mathrm{EFIR}}\right)^{-\delta}, \quad \theta \to \infty.$$

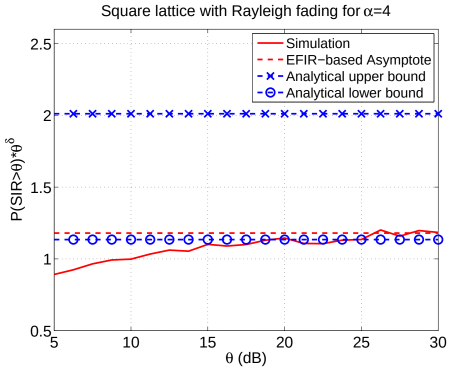
Corollary

$$G_\infty = \frac{\mathrm{EFIR}}{\mathrm{EFIR}_{\mathrm{PPP}}} = \left(\frac{\lambda\pi\mathbb{E}_o^!(I_\infty^{-\delta})\mathbb{E}(h^\delta)}{\operatorname{sinc}\delta}\right)^{1/\delta}.$$

Implication on tail of SIR distribution

The asymptotic behavior $p_s(\theta) = \Theta(\theta^{-\delta})$ is unavoidable for the singular path loss law and stationary BS deployment.

Scaled success probability $p_s(\theta)\theta^\delta$ for square lattice



The curve approaches EFIR^δ. The EFIR is bounded as

$$\frac{(\pi\Gamma(1+\delta))^{1/\delta}}{Z(2/\delta)} \leq \text{EFIR}_{\text{sq}} \leq \left(\frac{\pi}{\operatorname{sinc}\delta}\right)^{1/\delta} \frac{1}{Z(2/\delta)},$$

where Z is the Epstein zeta function. The asymptote is at $\sqrt{\text{EFIR}} \approx 1.19$.

Summary: MISR and EFIR

For $\theta \to 0$ and Rayleigh fading:

$$p_s(\theta) \sim 1 - \theta\,\mathrm{MISR}\,; \qquad G_0 = \frac{\mathrm{MISR}_{\mathrm{PPP}}}{\mathrm{MISR}}$$

For $\theta \to \infty$ and arbitrary fading:

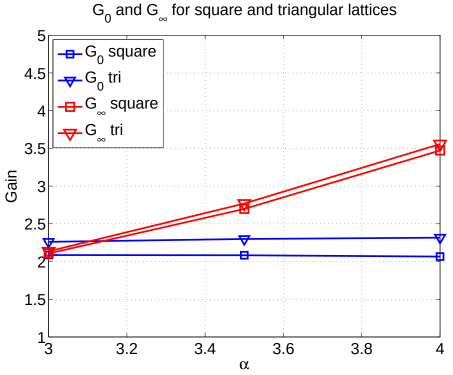
$$p_s(\theta) \sim \left(\frac{\theta}{\mathrm{EFIR}}\right)^{-\delta}; \qquad G_\infty = \frac{\mathrm{EFIR}}{\mathrm{EFIR}_{\mathrm{PPP}}}$$

The reference MISR and EFIR for the PPP have very simple expressions:

$$\mathrm{MISR}_{\mathrm{PPP}} = \frac{\delta}{1-\delta}\,; \qquad \mathrm{EFIR}_{\mathrm{PPP}} = (\operatorname{sinc}\delta)^{1/\delta}$$

They are efficiently obtained by simulation for arbitrary point processes.

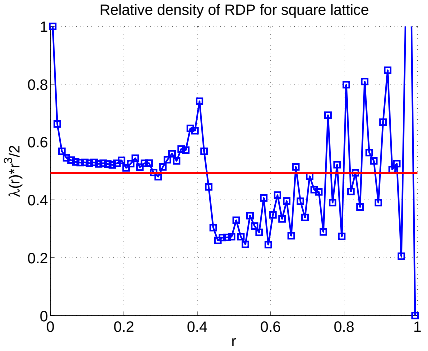
Asymptotic gains for lattices



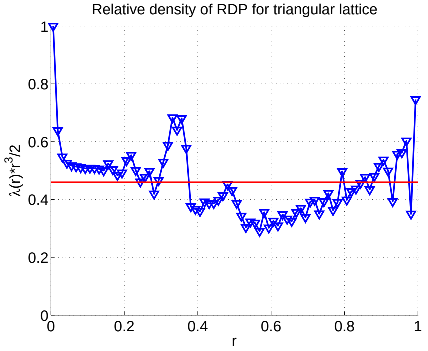
G_0 barely depends on α, while G_∞ slightly increases.

Insensitivity of G_0 to α

Recall: $\mathrm{MISR} = \int_0^1 r^\alpha \lambda(r)\mathrm{d}r$, where λ is the intensity function of the RDP.



$\lambda_{\text{sq}}(r)/\lambda_{\text{PPP}}(r)$

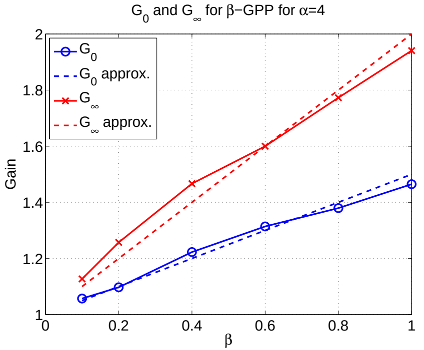


$\lambda_{\text{tri}}(r)/\lambda_{\text{PPP}}(r)$

Relative intensity of RDPs of square and triangular lattices.

The straight line corresponds to $1/G_{0,\text{sq}}$ and $1/G_{0,\text{tri}}$. It is essentially the average of the relative densities.

The gains for the β-Ginibre process

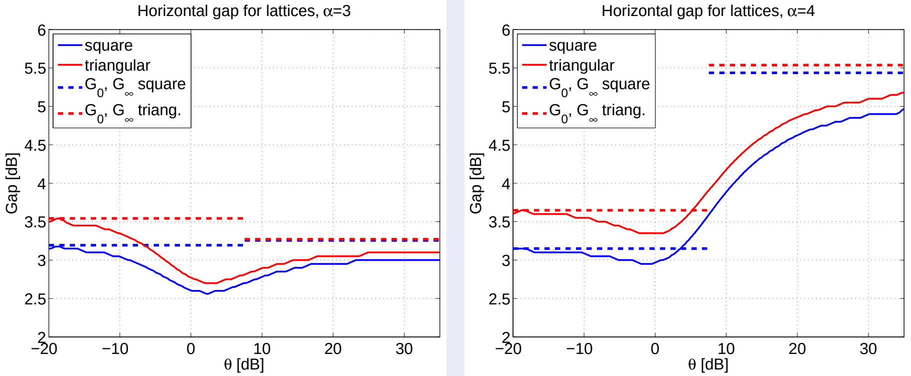


So quite exactly (and almost independently of α):

$$G_0(\beta) \approx 1 + \beta/2; \qquad G_\infty(\beta) \approx 1 + \beta.$$

The square lattice has gains of 2 and 3.5, so the 1-GPP falls quite exactly in between the PPP and the square lattice, both for G_0 and G_∞.

Gain trajectories $G(\theta)$ and asymptotics for lattices



The gap is relatively constant over more than 5 orders of magnitude for θ.

It is not monotonic, but probably $G(\theta) \leq \max\{G_0, G_\infty\}$.

Conclusions

- The world outside Poissonia is harsh. Even for the PPP, the SIR ccdfs for advanced transmission techniques (including MIMO) are unwieldy.
- To explain the unreasonable effectiveness of the ASAPPP method

$$p_{\mathrm{s}}(\theta) \approx p_{\mathrm{s,PPP}}(\theta/G_0),$$

  we have compared G_0 with G_∞, which is the gap at $\theta \to \infty$.
- The asymptotic gains G_0 and G_∞ are given by the MISR and the EFIR, respectively. The MISR is closely related to the relative distance process and can be generalized for different types of fading.
- G_0 and G_∞ are insensitive to fading, and G_0 is insensitive to α.
- The ASAPPP method is relatively accurate over the entire range of θ and highly accurate for $p_{\mathrm{s}}(\theta) > 3/4$ (or $\theta < 10$).
- A lot more work can and needs to be done.

Homework

The problem

Find a point process on W such that

$$g(x,y) \triangleq \frac{\rho^{(2)}(x,y)}{\lambda(x)\lambda(y)} = 2 \quad \forall x,y \in W_{\mathrm{int}}.$$

A solution

In the process of analyzing cellular networks, we have solved the homework problem...

Homework

The problem

Find a point process on W such that

$$g(x,y) \triangleq \frac{\rho^{(2)}(x,y)}{\lambda(x)\lambda(y)} = 2 \quad \forall x,y \in W_{\text{int}}.$$

A solution

The RDP of the PPP is a solution for $W = [0,1]$.
From slide 31, the second moment density is

$$\rho^{(2)}(x,y) = 8x^{-3}y^{-3}.$$

This is exactly $2\lambda(x)\lambda(y)$ since the intensity function is $\lambda(x) = 2x^{-3}$.

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