When do wireless network signals appear Poisson?

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Mathematical model of signals

With a "typical user" located at the origin, the model has three components:

1. Transmitter positions : $\{x_i\}_{i \in \mathbb{N}} \subseteq \mathbb{R}^2/\{0\}$.
2. A path-loss function: $\ell : \mathbb{R}^2/\{0\} \to (0, \infty)$.
3. Sequence of i.i.d. random variables $0 < S_1, S_2, \ldots$ representing shadowing and/or fading effects (due to eg signals colliding with obstacles).

Signal propagation model:

$$P_i = S_i \ell(x_i) = \frac{S_i}{g(x_i)}$$

where $g(x_i) := 1/\ell(x_i)$ is the path-gain function .

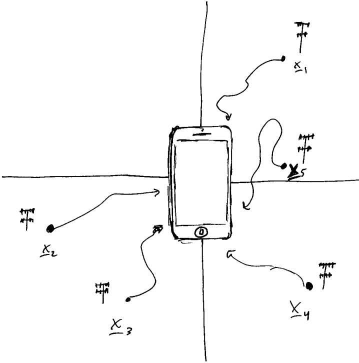

Figure : Sketch by N. Ross

What is the random behaviour of power strengths $\{P_1, P_2, \ldots\}$ or, equivalently, $\{1/P_1, 1/P_2, \ldots\}$?

Poisson transmitters implies Poisson signals

- Transmitters form a Poisson process $\Phi = \{X_i\}$ on $\mathbb{R}^2$ with density λ
- Define propagation or path loss process (inverse of power values P_i):

$$Z := \{Y_i\} \equiv \left\{ \frac{g(X_i)}{S_i} : X_i \in \Phi \right\}. \quad (1)$$

- Definition based on convention ie the strongest signals are near zero
- Captures how the network "appear" to a user.

Lemma (Just the displacement theorem)

Under the Poisson model with function $g(x) = |x|^\beta$ and random S such that $\mathbb{E}[S^{\frac{2}{\beta}}] < \infty$. Then the propagation process $Z = \{Y_i\}$ is an inhomogeneous Poisson point process on $\mathbb{R}_+$ with mean (or intensity) measure

$$\Lambda_Z([0, t)) = at^{\frac{2}{\beta}}$$

where $a := \lambda\pi\mathbb{E}(S^{\frac{2}{\beta}})$.
For general increasing g, the Poisson process Z has mean measure

$$\Lambda_Z([0, t)) = \lambda\pi\mathbb{E}[g(tS)^{-1}]^2$$

Deterministic positioning of transmitters

- For $0 < \lambda < \infty$, assume a deterministic point pattern $\phi = \{x_i\}_i \subseteq \mathbb{R}^2/\{0\}$ of transmitters such that

$$\frac{\phi(r)}{\pi r^2} \to \lambda, \quad \text{as} \quad r \to \infty.$$

  where $\phi(r)$ denotes the number of points of ϕ within distance r of the origin ie number of points of ϕ in $B_0(r)$.
- Assume (rescaled) log-normal fading variables:

$$S_i^{(\sigma)} = e^{\sigma N_i - \sigma^2/\beta},$$

  where N_i are i.i.d. standard normal variables.
- Assume $g(x) = |x|^\beta$.
- Propagation process:

$$W^{(\sigma)} := \left\{ \frac{g(x_i)}{S_i^{(\sigma)}} : x_i \in \phi \right\} = \left\{ \frac{|x_i|^{\beta_i}}{S_i^{(\sigma)}} : x_i \in \phi \right\}.$$

Signals can "appear" Poisson under strong fading

Theorem (Blaszczyszyn, Karray, Keeler 2013, 2014)

Provided $g(x) = |x|^\beta$ *and log-normal* $S_i^{(\sigma)}$ *, then as* $\sigma \to \infty$ *(implying* $S_i^{(\sigma)} \to 0$ *in distribution), the point process* $W^{(\sigma)} = \{Y_i^{(\sigma)}\}$ *converges weakly to an inhomogeneous Poisson point process on* $\mathbb{R}_+$ *with mean measure*

$$\Lambda_W([0, t)) = at^{\frac{2}{\beta}}$$

where $a := \lambda\pi\mathbb{E}([S_i^{(\sigma)}]^{\frac{2}{\beta}})$.

- Observed independently a couple of times via simulation eg Brown (2000), Błaszczyszyn and Karray (2012).
- Proof uses classic translation convergence results eg Chapter 11 in Daley and Vere-Jones (2008).
- Relies heavily upon properties of $g(x) = |x|^\beta$ and log-normal S_i eg normal distribution is divisible and symmetric, $g^{-1}(S_i)$ is also log-normal.
- Can this convergence result be extended to more general $g(x)$ and S_i?
- Can bounds be derived between $W^{(\sigma)}$ and a Poisson process with the same mean measure?

Assumptions and notation for generalization

Transmitter positioning:

Let $\phi = \{x_i\}_i \subseteq \mathbb{R}^d / \{0\}$ be a locally finite collection of points in $\mathbb{R}^d$ such that

$$\frac{\phi(r)}{\pi r^2} \to \lambda, \quad \text{as} \quad r \to \infty. \tag{2}$$

Define $\mathcal{I}$ as a finite or countable index set such that $\phi = \{x_i : \ i \in \mathcal{I}\}$.

Path-gain function:

Let $g : \mathbb{R}^d \to \mathbb{R}_+$ be a positive Borel measurable function.

Fading variables:

Let $\{S_i : \ i \in \mathcal{I}\}$ be a sequence of i.d.d. positive random variables.

Propagation process:

Let $Y_i = g(x_i)/S_i$ and define the corresponding propagation point process

$$W := \{Y_i\}_{i \in \mathcal{I}}$$

Let $p_i(t) := \mathbb{P}(0 < Y_i \leq t)$ and $M(t) := \sum_{i \in \mathcal{I}} p_i(t)$ be the mean measure of W
Let Z be a Poisson process on $\mathbb{R}_+$ having a mean measure $M(t)$.

Approximation theorem: Bounds on total variation

- Recall the total variation distance between two probability measures ν_1, ν_2 on the same probability space $(\mathcal{D}, \mathcal{F}(\mathcal{D}))$ is defined as

$$d_{\mathrm{TV}}(\nu_1, \nu_2) = \sup_{A \in \mathcal{B}(\mathcal{D})} |\nu_1(A) - \nu_2(A)|.$$

- Consider propagation point process W and Z restricted to compact domain $[0, t]$, denoted by $W|_t$ and $Z|_t$
- Denote the laws of $W|_t$ and $Z|_t$ by $\mathcal{L}(W|_t)$ and $\mathcal{L}(W|_t)$.

Theorem (Keeler, Ross, and Xia 2014)

Provided the previous conditions, then the following bounds hold

$$\frac{1 \wedge M(t)^{-1}}{32} \sum_{i \in \mathcal{I}} p_i(t)^2 \leq d_{TV}(\mathcal{L}(Z|_t), \mathcal{L}(W|_t)) \leq \sum_{i \in \mathcal{I}} p_i(t)^2 \leq M(t) \sup_{i \in \mathcal{I}} p_i(t).$$

- $\sum_i p_i(t)^2$ term is due to a coupling argument (cf Le Cam's theorem).
- Far RHS is from the definition of the mean measure $M(t) = \sum_i p_i(t)$; $\sup_{i \in \mathcal{I}} p_i(t)$ is based on the nearest point to the origin.
- Far LHS is due to Stein's method by Barbour and Hall (1984).
- Bounds hold for functions of point processes W and Z (eg strongest signal) which needs to be explored more.

Convergence theorem

Theorem (Keeler, Ross, and Xia 2014)

$g : \mathbb{R}^d \to \mathbb{R}_+$ such that $g(x) = h(|x|)$ for a continuous and nondecreasing h .
$(S(\sigma))_{\sigma \geq 0}$ is a family of positive random variables indexed by some non-negative parameter σ.
$W^{(\sigma)}$ is the process generated by $S(\sigma)$, g and ϕ .
If

$$(i)\ S(\sigma) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad (ii)\ \mathbb{E}[W^{(\sigma)}(t)] \to L(t),$$

as $\sigma \to \infty$, then $W^{(\sigma)}$ converges weakly to a Poisson process on $\mathbb{R}_+$ with mean measure L.

- Intuitively, most points of ϕ are being sent out to infinity in $W^{(\sigma)}$ because $S(\sigma)$ tends to zero, .
- Poisson limit is due to the thinning of the points in ϕ, but the retained points are redistributed
- Thinning scheme is different from the classical thinning schemes in the literature eg Kallenberg (1975), Brown (1979), Schuhmacher (2005, 2009).

Random positioning of transmitters

- Replace ϕ with a locally finite point process Φ independent of $\{S_i\}_{i\in\mathbb{N}}$.
- Define

$$M^{\Phi}(t) = \int_{\mathbb{R}^d} p_{(x)}(t)\Phi(dx),$$

where $p_{(x)}(t) = \mathbb{P}(0 < g(x)/S \leq t)$.

- Conditional on Φ, let Z be the Cox process directed by the measure M^{Φ}.

Theorem (Keeler, Ross, and Xia 2014)

For Φ, the following bounds hold

$$d_{TV}(\mathcal{L}(Z|_t), \mathcal{L}(W|_t)) \leq \mathbb{E}\int_{\mathbb{R}^d} p_{(x)}(t)^2\Phi(dx)$$

- For random Φ, an analogue of the previous convergence result is possible.
- Process may converge to a Cox process if the limit of its mean measure is random.
- When will it converge to a Poisson or Cox process?

Theorem (Keeler, Ross, and Xia 2014)

Assume that Φ *is a process on* $\mathbb{R}^d$ *with a locally finite mean measure* $\Lambda(r) := \mathbb{E}[\Phi(r)]$ *such that as* $r \to \infty$,

$$\Lambda(r) \to \infty, \qquad \frac{\mathrm{Var}(\Phi(r))}{\Lambda(r)^2} \to 0. \tag{3}$$

Assume i.i.d. $S_i(\sigma)$ *and* $g(x) = h(|x|)$, *where* h *is continuous, nondecreasing and positive on* $\mathbb{R}_+$. *If*

$$(i) S(\sigma) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad (ii) \int_{\mathbb{R}^d} \mathbb{P}\left(0 < \frac{g(x)}{S(\sigma)} \leq t\right) \Lambda(dx) \to L(t), \tag{4}$$

as $\sigma \to \infty$, *then* $W^{(\sigma)}$ *converges weakly to a Poisson process* Z^L *with mean measure* L.

(i) If $\Phi = \phi$ is non-random, $\mathrm{Var}(\Phi(r)) = 0$, then, again, the limit process is Poisson.

(ii) If Φ is a homogeneous Poisson process with parameter λ, then $\mathrm{Var}(\Phi(r))/(\mathbb{E}\Phi(r))^2 = (\lambda\pi r^2)^{-1}$ which tends to zero as $r \to \infty$, so the limit process is Poisson.

(iii) If Φ is a Cox process having intensity $\lambda_i > 0$ with probability $1/2$ for $i = 1, 2$ and $\lambda_1 \neq \lambda_2$, then the limit process is a Cox process directed by a random mean measure that takes two measures with equal probability. We can derive that

$$\mathbb{E}\Phi(r) = \frac{\lambda_1 + \lambda_2}{2}\pi r^2, \qquad \mathrm{Var}(\Phi(r)) = \frac{\lambda_1 + \lambda_2}{2}\pi r^2 + \frac{(\lambda_1 - \lambda_2)^2}{4}\pi^2 r^4,$$

and hence $\lim_{r\to\infty} \Lambda(r) = \infty$, but

$$\lim_{r\to\infty} \frac{\mathrm{Var}(\Phi(r))}{(\mathbb{E}\Phi(r))^2} = \frac{(\lambda_1 - \lambda_2)^2}{(\lambda_1 + \lambda_2)^2} \neq 0.$$

Cox process via a (dependent) product fading-shadowing model

Transmitters are placed on $\mathbb{R}^2$, $g(x) = (|x|)^\beta$ for $\beta > 2$, and $\phi(r)/r^2 \to \lambda\pi$.
Suzuki product fading-shadowing model:

$$S_i(\sigma) = \exp\{\sigma N_i - \sigma^2/\beta\} F_i,$$

where $N_1, N_2, \ldots$ are i.i.d. standard normal random variables independent of F_i, and each F_i is exponential such that $\mathbb{E} F_i^{2/\beta} = 1$ and $\mathbb{E} S_i^{2/\beta}(\sigma) = 1$.
$W^{(\sigma)}$ converges to Poisson process with mean measure

$$M(t) = \lambda\pi t^{2/\beta}.$$

Replace i.i.d. F_i with a common F variable (not necessarily exponential).
Modified product fading-shadowing model:

$$S_i^F(\sigma) = \exp\{\sigma N_i - \sigma^2/\beta\} F$$

Then conditional on F , $W^{(\sigma)}$ converges to a Process process with a random mean measure

$$M^F(t) := F^{2/\beta} \lambda\pi t^{2/\beta}.$$

$W^{(\sigma)}$ converges to a Cox process directed by M^F .

Interesting questions

- For a given transmitter configuration, do some fading models induce a propagation point process significantly closer to Poisson than others?
- How do the results translate to functions of the point process?
- What statistical parameter estimation methods can be developed?
- Can the results be extended to models with short range (spatial) dependence between the fading variables? For example, model shadowing with Gaussian random fields.
- How can the results be generalized and applied to other settings?

Thank you.

References:

T. X. Brown. *Cellular performance bounds via shotgun cellular systems*, 2000

B. Błaszczyszyn, H.P. Keeler and M.K. Karray *Wireless networks appear Poissonian due to strong shadowing*, to appear in IEEE Transactions on Wireless Communications, 2014

H.P. Keeler, N. Ross and A. Xia *When do wireless network signals appear Poisson?*, submitted, 2014

A simple example with Bernoulli fading variables (by N. Ross)

Consider a point pattern ϕ such that the mapped points $\{g(x_i)\}_i$ are the positive integers $\{1, 2, ...\}$.
Divide each point i by a random variable $S_i(\sigma)$, hence the point process

$$1/S_1(\sigma), 2/S_2(\sigma), ...,$$

where the $S_i(\sigma)$ are i.i.d. taking only two possible values:

$$P(S(\sigma) = 1/\sigma) = 1 - P(S(\sigma) = \sigma) = 1 - 1/\sigma.$$

$S(\sigma)$ tends to zero in probability as σ goes to infinity and by computing directly $P(i/S_i(\sigma) \leq t)$, we can see that the number of points in the interval $(0, t]$ converges to a Poisson variable, since

$$\mathbb{E}[\# \text{ of points} \leq t] = \sum_i P(i/S_i(\sigma) \leq t) \to t \quad \text{as} \quad n \to \infty,$$

and

$$\sum_i P(i/S_i(n) \leq t)^2 \to 0 \quad \text{as} \quad n \to \infty.$$

The previous theorem implies that the point process $\{1/S_1(\sigma), 2/S_2(\sigma), ...\}$ converges to a (homogeneous) Poisson process as σ goes to infinity.