Random sets in economics, finance and insurance

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based on joint works with
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A.Haier (Swiss Financial Market Supervision)

University Austin TX, May 2015

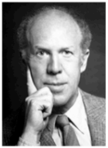
Early years

- Why do people make particular choices?
- How to allocate assets optimally between agents?

50 years and 3 Nobel Prizes

- Gerard Debreau (1983)
  (Debreau expectation of random sets)

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1983



Gerard Debreu

Prize share: 1/1

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1983 was awarded to Gerard Debreu *"for having incorporated new analytical methods into economic theory and for his rigorous reformulation of the theory of general equilibrium"*.

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50 years and 3 Nobel Prizes

- Robert Aumann and Thomas Schelling (2005) (Aumann expectation of random sets)

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005

Photo: D. Porges

Robert J. Aumann

Prize share: 1/2

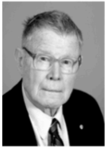

Photo: T. Zadig

Thomas C. Schelling

Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005 was awarded jointly to Robert J. Aumann and Thomas C. Schelling *"for having enhanced our understanding of conflict and cooperation through game-theory analysis"*

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50 years and 3 Nobel Prizes

- Alvin Roth and Llloyd Shapley (2012)
  (allocations: choice of an element of a random set)

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012

Photo: U. Montan
Alvin E. Roth
Prize share: 1/2

Photo: U. Montan
Lloyd S. Shapley
Prize share: 1/2

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2012 was awarded jointly to Alvin E. Roth and Lloyd S. Shapley *"for the theory of stable allocations and the practice of market design"*

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Overview

Developments over the last 10 years

Lecture 1

- Random sets and selections in economics.
- Market imperfections and transaction costs.

Lecture 2

- Sublinearity and risk.
- Financial networks.

- A. Beresteanu, I. Molchanov, and F. Molinari. Sharp identification regions in models with convex moment predictions. *Econometrica*, 79:1785–1821, 2011.
- I. Molchanov and I. Cascos. Multivariate risk measures: a constructive approach based on selections. *Math. Finance*, 2015.
- I. Molchanov and M. Schmutz. Multivariate extensions of put-call symmetry. *SIAM J. Financial Math.*, 1:396–426, 2010.
- Works in progress ...

Basics of random sets

Definition

A map $\mathbf{X} : \Omega \mapsto \mathcal{F}$ from a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ to the family of closed subsets of a Polish space $\mathfrak{X}$ is said to be a random closed set if

$$\{\omega : \mathbf{X}(\omega) \cap G \neq \emptyset\} \in \mathfrak{F}$$

for all open sets G.

- In $\mathfrak{X} = \mathbb{R}^d$ one can take compact sets K instead of open G.

Examples

So far most of examples involve "simple" sets scattered in space.

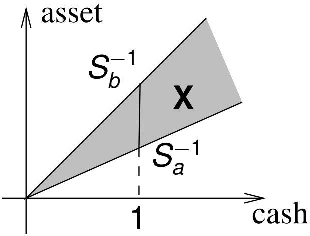
- Point process.
- Random geometric graphs.
- Collection of lines/balls, etc.
- Random polytopes.

Examples: Economics

- Consider a game of d players with random parameters. Then the set of equilibria (pure or mixed) is a subset of the unit cube in $\mathbb{R}^d$.
- Measurements are often represented as intervals rather than points. The reason may be not only the lack of precision or censoring, but also intentional reporting of intervals.
- If the underlying probability measure $\mathbf{P}$ is uncertain, this uncertainty can be described as a family of random elements.

Examples: Finance

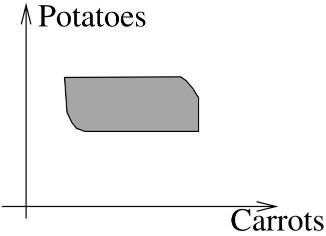
- Each asset has two prices $S_b \leq S_a$: bid and ask prices.
- Adding cash as one axis, this leads to a random cone.



- Multiasset setting leads to more complicated random cones. Example: currency exchanges with transaction costs (Kabanov's model).

Examples: Finance (link-save)

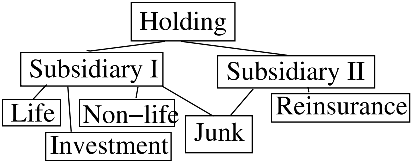
- Buying carrots and potatoes together brings a reduction.



Example: Financial networks

Here is a picture
of a large network
connecting major financial
institutions in the world

Example: Financial network



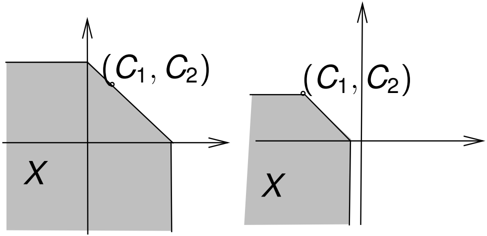
- Objective: IntraGroup Transfers (in order to help distressed members of the group)

Example: Financial network, two agents

- Two agents A and B are assessed by a regulator.
- The regulator evaluates their individual exposure to risks and requests that they set aside (and freeze) necessary capital reserves.
- The agents want to minimise these reserves and conclude an agreement that at the terminal time (and under certain conditions) one of them would help to offset the deficit of another one.
- The family of allowed transactions is a random closed set in the plane.

Examples: financial network, two agents

- The terminal capital is (C_1, C_2).
- Transfers from a solvent company to another one are allowed up to the available positive capital.
- Disposal of assets is allowed.



Selections

- A random vector ξ is called a selection of **X** if both ξ and **X** can be realised on the same probability space, so that $\xi \in \mathbf{X}$ a.s.
- From now on we tacitly assume that all random elements are realised on the same probability space.

Theorem (Zvi Artstein, 1983)

A probability measure μ is the distribution of a selection of a random closed set **X** *in $\mathbb{R}^d$ if and only if*

$$\mu(K) \leq \mathbf{P}\{\mathbf{X} \cap K \neq \emptyset\}$$

for all compact sets $K \subset \mathbb{R}^d$.

Games and payoffs

- Set K is a coalition of players.
- $\varphi(K)$ is the payoff that K receives.
- Payoff functional is not additive, but subadditive.
- Allocation is a measure μ such that

$$\mu(K) \leq \varphi(K) \qquad \forall\, K.$$

- Bondareva–Shapley theorem: existence of an allocation under some conditions on φ (convexity).

Example: market entry game (non-coalitional)

- Payoff for the jth player, $j = 1, 2$,

$$\pi_j = a_j(a_{-j}\theta_j + \varepsilon_j)\,,$$

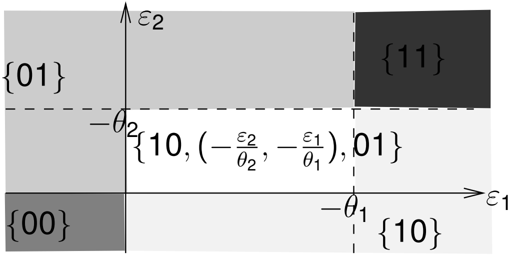
where

  - $a_j \in \{0, 1\}$ is the action (enter or not) of the jth player;
  - a_{-j} is the action of other player(s),
  - θ_j are unknown parameters
  - ε_j are random profit shifters with known distribution

Example: market entry game — equilibria

$$\pi_j = a_j(a_{-j}\theta_j + \varepsilon_j), \quad j = 1, 2.$$

- Set $S_\theta(\varepsilon)$ of (Nash) equilibria is random and depends on ε.



Note that $\theta_1, \theta_2 < 0$.

Example: market entry game - inference

- Assume pure strategies only
  (the method works also for mixed strategies).
- The econometrician observes empirical variant of the distribution

$$\mu = (p_{00}, p_{10}, p_{01}, p_{11}).$$

- These frequencies are sampled from the set of possible equilibria, i.e.

$$\mu \text{ is the distribution of a selection of } S_\theta(\varepsilon).$$

Inference

Estimate $\theta = (\theta_1, \theta_2)$ based on this, i.e. estimate parameters of a random set by observing its selection:

$$\Big\{\theta : \ \mu(K) \le \mathbf{P}\{S_\theta \cap K \neq \emptyset\} \quad \forall K \subset \{00, 01, 10, 11\}\Big\}.$$

The most serious difficulty

- The family of selections is too rich.
- It is numerically difficult to verify inequalities $\mu(K) \leq \mathbf{P}\{\mathbf{X} \cap K \neq \emptyset\}$ for all K.

Course exercise

- ξ has normal distribution $N(\mu, \sigma^2)$.
- Observe numbers $x_1, \dots, x_n$ chosen (using an unknown mechanism) such that $x_i \geq \xi_i$ for i.i.d. realisations $\xi_1, \dots, \xi_n$ of ξ.
- Estimate μ and σ^2 and utilise the available information in full!

Castaing representation

Theorem (Charles Castaing, 1977)

X *is a random closed set if and only if*

$$\mathbf{X} = \mathrm{cl}(\{\xi_n, n \geq 1\})$$

meaning that **X** *is the closure of a countable family of its selections.*

Definition

$L^p(\mathbf{X})$ denotes the family of p-integrable selections of **X**.

Expectations of selections

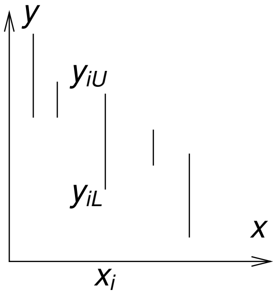
- Assume that $\mathbf{X}$ has at least one integrable selection, i.e. $L^1(\mathbf{X}) \neq \emptyset$.
- The (Aumann) expectation of $\mathbf{X}$

$$\mathbf{E}\mathbf{X} = \mathrm{cl}\{\mathbf{E}\xi : \ \xi \in L^1(\mathbf{X})\}.$$

- The expectation is always a convex set if the probability space is non-atomic (follows from Lyapunov's theorem on range of a vector-valued measure).
- Higher moments are not well defined!

Example: interval least squares I

- Explanatory variable x
- Response $y \in Y = [y_L, y_U]$.



Example: interval least squares II

- Regression model (for mean values). If $y \in Y$ a.s., then

$$\mathbf{E}(y) = \theta_1 + \theta_2 \mathbf{E}(x)\,.$$

- Then

$$(\theta_1, \theta_2) = \Sigma(x)^{-1} \begin{bmatrix} \mathbf{E}(y) \\ \mathbf{E}(xy) \end{bmatrix}, \qquad \Sigma(x) = \begin{bmatrix} 1 & \mathbf{E}x \\ \mathbf{E}x & \mathbf{E}x^2 \end{bmatrix}.$$

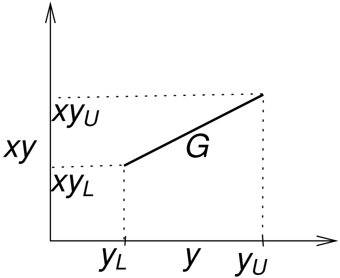
- The expectations $\mathbf{E}(y)$ and $\mathbf{E}(xy)$ may take various values depending on the choice of selection $y \in Y = [y_L, y_U]$.

Example: interval least squares III

- y is a selection of Y and xy is a selection of xY
- Thus, (y, xy) is a selection of

$$G = \left\{ \begin{pmatrix} y \\ xy \end{pmatrix} : \ y_L \le y \le y_U \right\} \subset \mathbb{R}^2$$

segment with end-points (y_L, xy_L) and (y_U, xy_U).



- Identification region $\theta \in \Sigma(x)^{-1}\mathbf{E}G$.

Course exercises

1. How to amend the setting for the polynomial regression?
2. How to handle the case of interval-valued explanatory variable x?

Options (main idea)

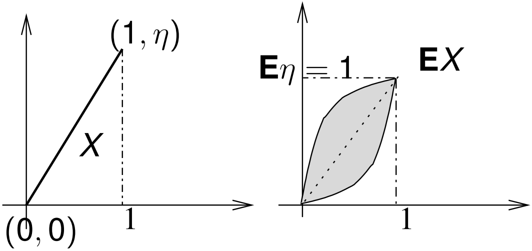
- Call price

$$\mathbf{E}_{\mathbf{Q}}(F\eta - k)_+$$

and put price

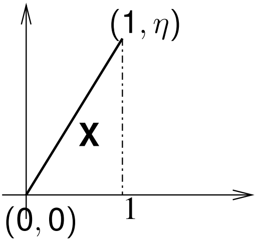
$$\mathbf{E}_{\mathbf{Q}}(k - F\eta)_+$$

can be expressed in terms of the expectation of the random set **X**.



Option prices I

- Let η be a non-negative random variable (relative price change).



- The support function of **X** in direction u is

$$h_{\mathbf{X}}(u) = \sup\{\langle u, x\rangle : \ x \in \mathbf{X}\} = (u_1 + u_2\eta)_+$$

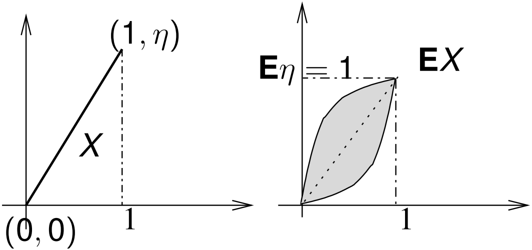
Option prices II

$$h_{\mathbf{X}}(u) = (u_1 + u_2\eta)_+$$

- If $u = (-k, F)$, then $h_{\mathbf{X}}(u) = (F\eta - k)_+$ is the payoff from the call option.
- In this case:
  - F is the forward price (deterministic carrying costs);
  - $S = F\eta$ is the terminal price;
  - k is the strike price (buying price at the terminal time).
- If $u = (k, -F)$, then $h_{\mathbf{X}}(u) = (k - F\eta)_+$ is the payoff from the put option.

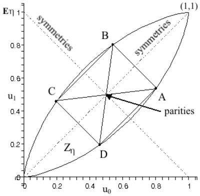
Option prices III

- The expected support function $\mathbf{E}h_{\mathbf{X}}(u)$ is the support function of the expectation $h_{\mathbf{EX}}(u)$.
- Then $h_{\mathbf{EX}}(-k, F) = \mathbf{E}_{\mathbf{Q}}(F\eta - k)_+$ is the call price if the expectation is taken with respect to the martingale measure.



Symmetries

- Point symmetry with respect to $(\frac{1}{2}, \frac{1}{2})$ is equivalent to European put-call parity



- Line symmetry is equivalent to put-call symmetry.

Multi-asset symmetry

Asset prices $S_{T1} = F_1\eta_1, \ldots, S_{Td} = F_d\eta_d$

Prices of basket options

$$\mathbf{E_Q}(u_0 + u_1\eta_1 + \cdots + u_d\eta_d)_+$$

(forward prices are included in the weights).

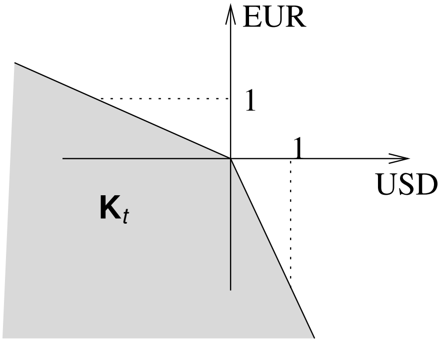
- When is the price invariant for all permutations of the weights (self-duality)?
- If this is the case, then η is exchangeable.
- However, the exchangeability property does not suffice, e.g. η with i.i.d. coordinates.
- When is the price invariant for $u_0 = 0$ and permutations of other weights (swap-invariance)?

Convex models of transaction costs

- Let $(\Omega, \mathfrak{F}_t, t = 0, \ldots, T, \mathbf{P})$ be a stochastic basis.
- Let $\mathbf{K}_t$, $t = 0, \ldots, T$, be a sequence of random convex sets so that $\mathbf{K}_t$ is $\mathfrak{F}_t$-measurable.
- Sets $\mathbf{K}_t$ contain the origin and are lower sets. Assume that they do not contain any line.
- Set $\mathbf{K}_t$ describes the positions available at price zero at time t.

Kabanov's exchange cone model

- $\mathbf{K}_t$ is a random exchange cone (e.g. generated by bid-ask exchange rates for currencies at time t).
- $\mathbf{K}_t$ is the family of portfolios available at price zero.
- Reflected set $-\mathbf{K}_t$ is the family of solvent positions at time t.



Self-financing

- A self-financing portfolio process satisfies

$$V_t - V_{t-1} \in L^0(\mathbf{K}_t, \mathfrak{F}_t)$$

  so the increment is available at price zero.
- The set

$$\mathcal{A}_t = \sum_{i=0}^{t} L^0(\mathbf{K}_i, \mathfrak{F}_i) \subset L^0(\mathbb{R}^d)$$

  is the family of attainable claims at time t.

No arbitrage

Definition

(NAS) (strict no-arbitrage) condition holds if

$$\mathcal{A}_t \cap L^0(-\mathbf{K}_t, \mathfrak{F}_t) = \{0\}$$

for all $t = 0, \ldots, T$.

Interpretation

Starting from zero it is not possible to achieve non-zero solvent position at any time t.

No arbitrage and martingales (cone models)

- Define the dual cone

$$\mathbf{K}_t^* = \{u : \langle u, x \rangle \leq 0 \quad \forall\, x \in \mathbf{K}_t\}$$

- It describes the family of consistent price systems, if u are prices, then each portfolio available at price zero indeed has negative price.

Theorem (Kabanov et al.)

(NAS) condition is equivalent to the existence of an equivalent probability measure $\mathbf{Q}$ *and a* $\mathbf{Q}$*-martingale* M_t *such that*

$$M_t \in \text{relative interior } \mathbf{K}_t^*, \qquad t = 0, \dots, T.$$

Conditional expectation and martingales

- A sequence of random sets $\mathbf{X}_t$, $t = 0, \dots, T$, is a martingale if

$$\mathbf{E}(\mathbf{X}_t | \mathfrak{F}_s) = \mathbf{X}_s \quad \text{a.s. } \forall\, s \le t.$$

- The conditional expectation is defined as the family of conditional expectations of all selections.

Cores and conditional cores

Definition

If **X** is $\mathfrak{F}$-measurable random closed set and $\mathfrak{A}$ is a sub-σ-algebra, then the conditional core

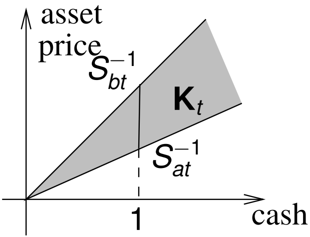
$$\mathbf{Y} = \mathbf{m}(\mathbf{X}|\mathfrak{A})$$

is the largest $\mathfrak{A}$-measurable random set **Y** such that $\mathbf{Y} \subset \mathbf{X}$ a.s.

- The conditional core exists.
- If $\mathbf{X} = (-\infty, \xi]$, then the conditinal core is the set $\mathbf{Y} = (-\infty, \eta]$, where η is the largest $\mathfrak{A}$-measurable random variable such that $\eta \leq \xi$.

Single asset case

- Recall the sequence of exchange sets $\mathbf{K}_t$ and price intervals $\mathbf{X}_t = [S_{bt}, S_{at}]$, $t = 0, \ldots, T$.
- (NAS) (no arbitrage) condition.



No arbitrage and conditional cores

Theorem

In case of a single asset with bid-ask spread $\mathbf{X}_t = [S_{bt}, S_{at}]$*, the (NAS) condition holds if and only if*

$$\mathbf{X}_t \subseteq \mathbf{m}(\mathbf{X}_{t+1}|\mathfrak{F}_t), \qquad t = 0, \ldots, T-1.$$