Random sets in economics, finance and insurance
Part II

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based on joint works with
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University Austin TX, May 2015

Non-linearity

- The capacity functional

$$T(K) = \mathbf{P}\{\mathbf{X} \cap K \neq \emptyset\}$$

determines the distribution of random closed set $\mathbf{X}$.
- T is not additive — it is subadditive.
- Further examples of such functionals appear as payoffs of coalitional games.
- Another source of such functionals is upper envelopes of measures:

$$\varphi(K) = \sup_{\mu \in \mathcal{M}} \mu(K)$$

Choquet integral

- Non-additive measure $\varphi(A)$ is a set function with values in $[0,1]$ such that $\varphi(\emptyset) = 0$.
- φ is called a capacity if it satisfies some continuity conditions.

Definition

If $f : \mathbb{R}^d \mapsto \mathbb{R}_+$, then

$$\int f d\varphi = \int_0^\infty \varphi(\{x : \ f(x) \geq t\}) dt$$

- If $\varphi(K) = T(K) = \mathbf{P}\{\mathbf{X} \cap K \neq \emptyset\}$, then

$$\int f dT = \mathbf{E} \sup f(\mathbf{X}) = \mathbf{E} \sup_{\xi \in \mathbf{X}} f(\xi) = \sup_{\xi \in \mathbf{X}} \mathbf{E} f(\xi).$$

Measuring risk

- Let ξ be the terminal financial position (gain if positive, loss if negative) of an agent.
- The regulator requires to set aside capital $r(\xi)$ as a security in case of possible eventual distress.

Univariate coherent L^p-risk measure

$r : L^p(\mathbb{R}) \mapsto (-\infty, \infty]$

- cash-invariant

$$r(\xi + a) = r(\xi) - a;$$

- monotonone

$$\xi \leq \eta \quad \Rightarrow \quad r(\xi) \geq r(\eta);$$

- subadditive

$$r(\xi + \eta) \leq r(\xi) + r(\eta);$$

- homogeneous

$$r(c\xi) = cr(\xi), \quad c > 0\,.$$

Examples

- Expectation
$$r(\xi) = -\mathbf{E}\xi$$
- Essential infimum
$$r(\xi) = -\text{ess inf}\,\xi.$$
- Expected Shortfall at level α
$$r(\xi) = -\mathbf{E}(\xi|\xi \le q_\alpha),$$
where q_α is the α-quantile of ξ (assume non-atomic distribution of ξ).
- Choquet integrals $r(\xi) = \int \xi d\varphi$ for an appropriate capacity on Ω.

Multivariate risks

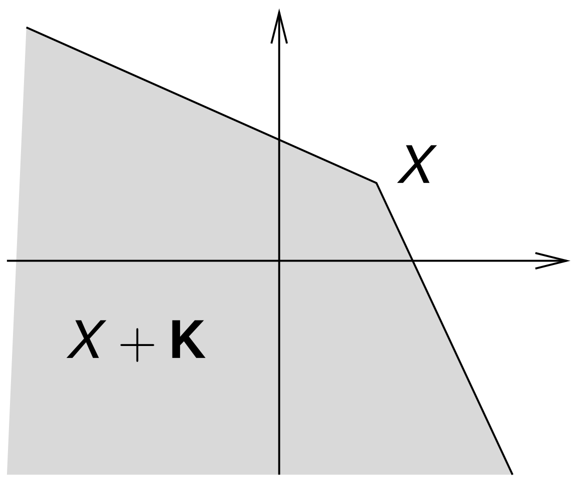
- The theory of risk measures started with the univariate case
- and then moved to the dynamic setting.
- The situation with multivariate risks only recently received a proper attention.

Kabanov's exchange cone model

$$\mathbf{X} = X + \mathbf{K}$$

- X is L^p-integrable random vector in $\mathbb{R}^d$ (gains on d assets/currencies).
- $\mathbf{K}$ is a random exchange cone (e.g. generated by bid-ask exchange rates for currencies).
- $\mathbf{K}$ is the family of portfolios available at price zero.
- $\mathbf{K}$ describes transaction rules at the time when the gain X is assessed.

Exchange cone

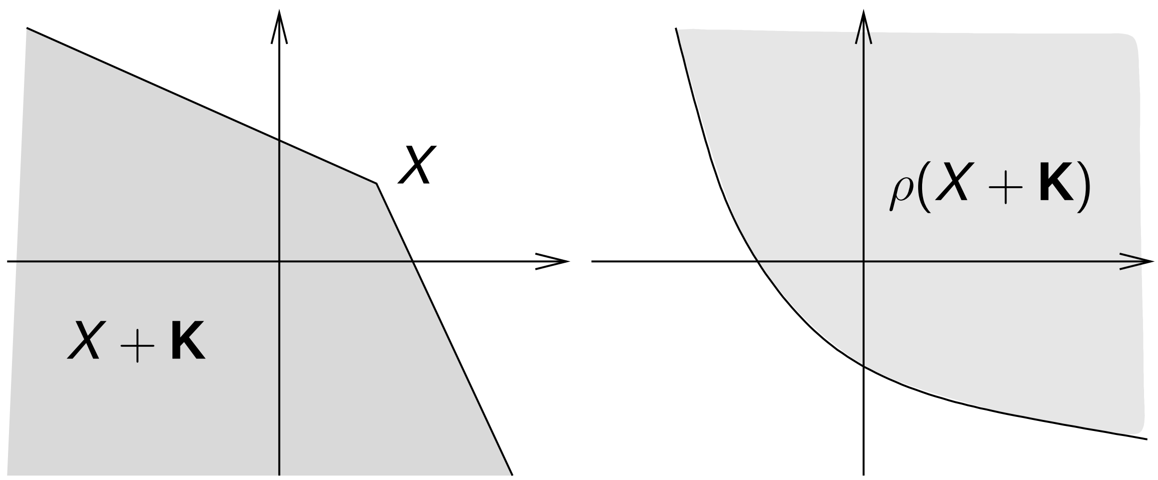


- Aim: measure the risk of X taking $\mathbf{K}$ into account.

Set-valued portfolio

- Idea: consider $\mathbf{X} = X + \mathbf{K}$ as a random set.
- Consider also other random sets $\mathbf{X}$ not necessarily of point-plus-cone type.
- Portfolio $\mathbf{X}$ is a random convex closed set such that $\mathbf{X} = \mathbf{X} + \mathbb{R}^d_-$ (lower set).
- Risk $\rho(\mathbf{X})$ is an upper set.

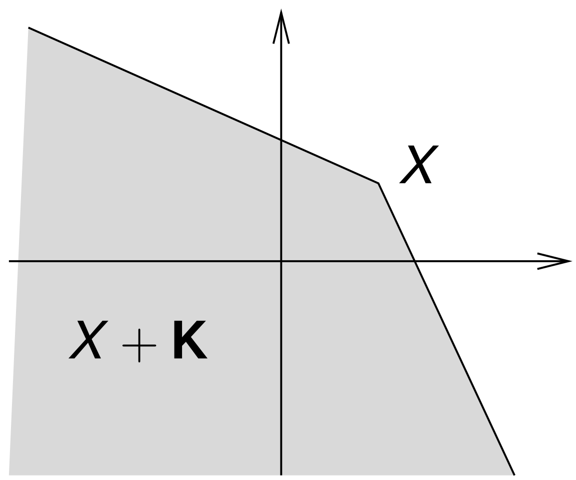
Portfolio and risk measure



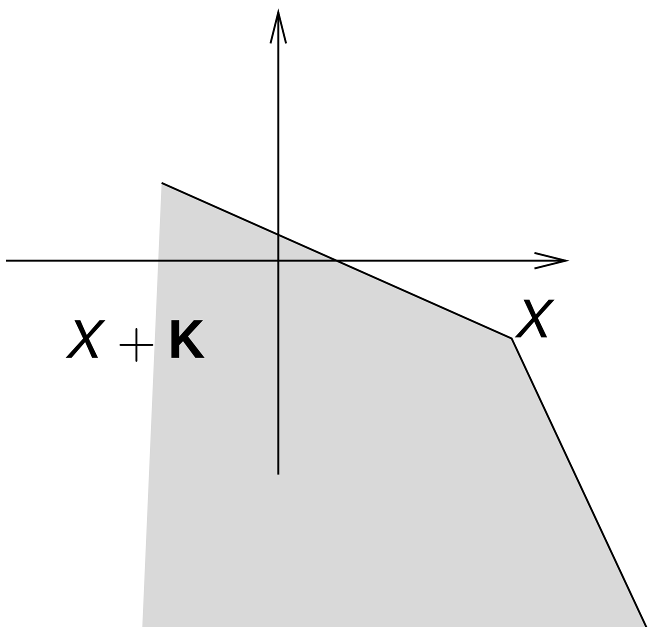
- Position is acceptable if the risk measure contains the origin.
- Larger set means lower risk.

Acceptability

- A portfolio **X** is acceptable if there exists a random vector
$$\xi \in \mathbf{X} \qquad \text{a.s.},$$
so that all components of ξ are individually acceptable.
- This means that there exists a terminal transfer that makes all lines acceptable.



Although components of ξ are considered separately, the ξ itself appears as linear combination of components of X and **K**, so that the results and not marginalised.



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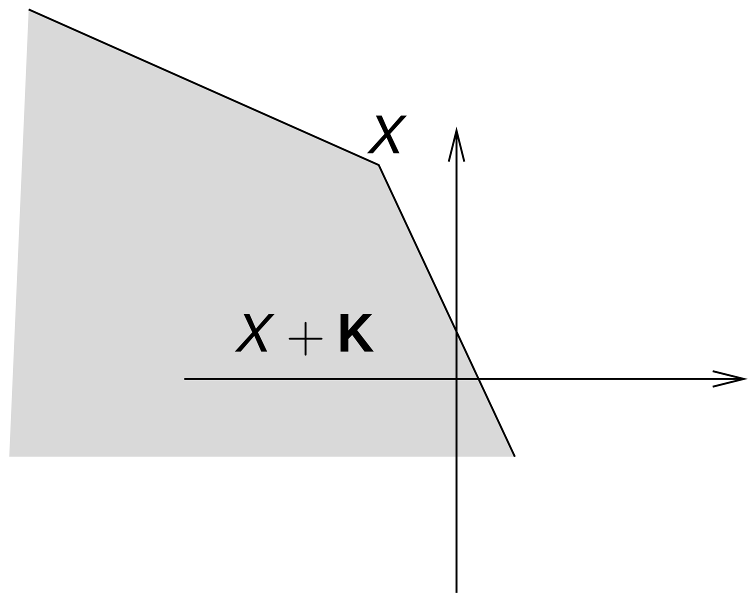
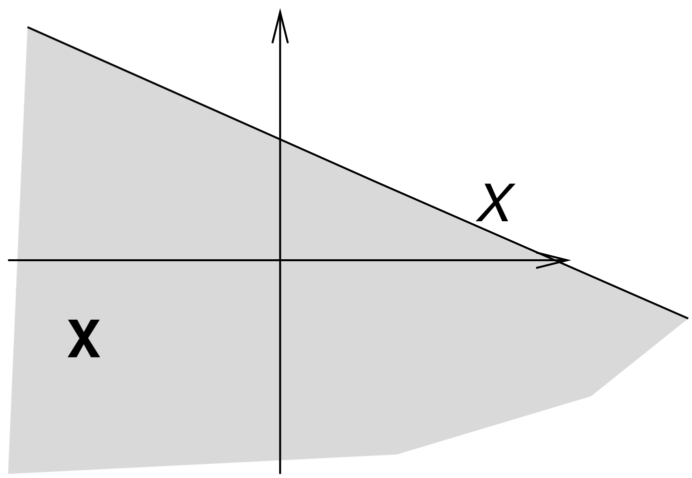


Although components of ξ are considered separately, the ξ itself appears as linear combination of components of X and **K**, so that the results and not marginalised.

Example

- Let $X = (X_1, X_2)$ be bivariate normal with means $(0.5, 0.5)$, variances $(1, 1)$ and correlation 0.6.
- Exchanges are free from transaction costs.
- Initial exchange rate $\pi_0 = 1.5$, terminal rate π is log-normal mean 1.5 and volatility 1.4, independent of X.
- Take Expected Shortfall at level 0.05 as the risk measure.

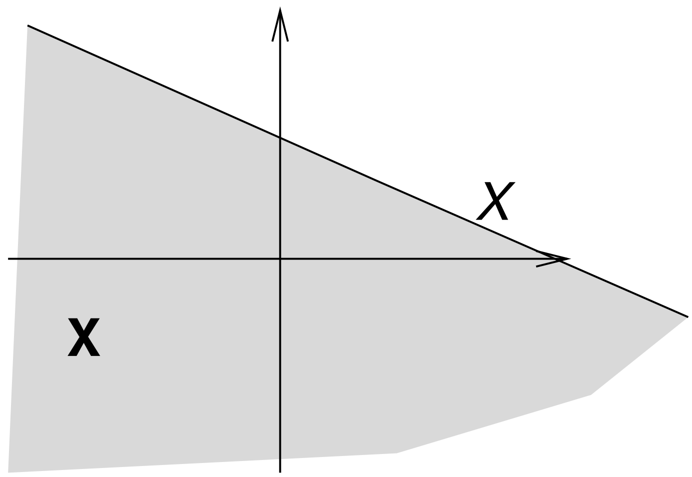
Example



The required capital in the first currency

No transfers	2.0645
Terminal transfer to the 1st currency	1.801
Terminal transfer to the 2nd currency	1.784
“Cleverer” transfer	1.661

Example



This "cleverer" transfer is given by

$$\xi = (\xi_1, \xi_2) = \left(\frac{\pi(X_1\pi + X_2)}{1+\pi^2}, \frac{X_1\pi + X_2}{1+\pi^2} \right)$$

Acceptable selections

- A random vector $\xi \in \mathbb{R}^d$ is called a selection of $\mathbf{X}$ if $\xi \in \mathbf{X}$ a.s.
  Assume that $\mathbf{X}$ contains at least one L^p-integrable selection, i.e. the set $L^p(\mathbf{X})$ is not empty.
- Consider d-tuple $\mathbf{r} = (r_1, \dots, r_d)$ of univariate coherent law invariant L^p-risk measures.
- Random set $\mathbf{X}$ is acceptable if it possesses at least one acceptable selection ξ, meaning that

$$\mathbf{r}(\xi) = (r_1(\xi_1), \dots, r_d(\xi_d)) \le 0\,,$$

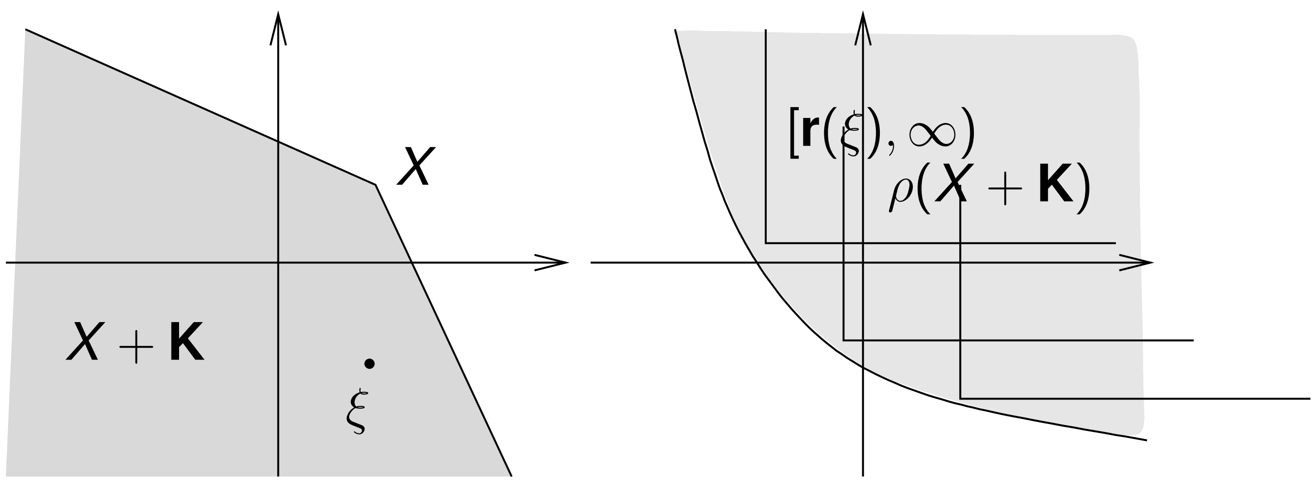
  i.e. all individual coordinates of ξ are acceptable.

Selection risk measure

- The selection risk measure $\rho_s(\mathbf{X})$ is the topological closure of the set

$$\rho_{s,0}(\mathbf{X}) = \{a \in \mathbb{R}^d : \mathbf{X} + a \text{ is acceptable}\} .$$

Portfolio and risk measure



The risk measure is the closed union of quadrants $\mathbf{r}(\xi) + \mathbb{R}^d_+$ determined by all possible selections ξ:

$$\rho_{\rm s}(\mathbf{X}) = \mathrm{cl} \bigcup_{\xi \in L^p(\mathbf{X})} [\mathbf{r}(\xi), \infty) \,.$$

Main properties

Theorem

The selection risk measure takes values being upper convex closed sets, is law invariant and is coherent, i.e. satisfies the following conditions

1. $\rho_s(\mathbf{X} + a) = \rho_s(\mathbf{X}) - a$ *for all* $a \in \mathbb{R}^d$ *(cash invariance).*
2. *If* $\mathbf{X} \subset \mathbf{Y}$ *a.s., then* $\rho_s(\mathbf{X}) \subset \rho_s(\mathbf{Y})$ *(monotonicity).*
3. $\rho_s(c\mathbf{X}) = c\rho_s(\mathbf{X})$ *for all* $c > 0$ *(homogeneity).*
4. $\rho_s(\mathbf{X} + \mathbf{Y}) \supset \rho_s(\mathbf{X}) + \rho_s(\mathbf{Y})$ *(superadditivity for inclusion = subadditivity of risks).*

Law invariance

- We assume that the probability space is non-atomic.
- Two identically distributed random sets **X** and **Y** might have rather different families of selections.
- Let $\mathfrak{F}_{\mathbf{X}}$ be the σ-algebra generated by **X**.
- If $\xi \in L^p(\mathbf{X})$ and $\mathbf{r}(\xi) \leq 0$, then

$$\mathbf{r}(\mathbf{E}(\xi|\mathfrak{F}_{\mathbf{X}})) \leq \mathbf{r}(\xi) \leq 0$$

  and $\eta = \mathbf{E}(\xi|\mathfrak{F}_{\mathbf{X}})$ is also a selection of **X**, since **X** is a.s. convex.
- Use the fact that the families of $\mathfrak{F}_{\mathbf{X}}$-measurable selections of **X** and $\mathfrak{F}_{\mathbf{Y}}$-measurable selections of **Y** coincide.

Selection (Aumann) expectation

- The closure of the set of expectations of all integrable selections
  $$\mathbf{E}\mathbf{X} = \mathrm{cl}\{\mathbf{E}\xi : \ \xi \in L^1(\mathbf{X})\}$$
  is called the selection (Aumann) expectation of $\mathbf{X}$.
- The closure is not needed if $\mathbf{X}$ is integrably bounded, i.e. $\|\mathbf{X}\| = \sup\{\|x\| : \ x \in \mathbf{X}\}$ is integrable.
- If X, Y are integrably bounded,
  $$\mathbf{E}(\mathbf{X} + \mathbf{Y}) = \mathbf{E}\mathbf{X} + \mathbf{E}\mathbf{Y}$$
- If $-\mathbf{E}\mathbf{X} = \{-x : \ x \in \mathbf{E}\mathbf{X}\}$, then
  $$\rho_{\mathrm{s}}(\mathbf{X}) = -\mathbf{E}\mathbf{X}$$
  is a (rather trivial) selection risk measure.

Capital reserving

- Let $\mathbf{X} = X + \mathbf{K}$ for a possibly random exchange cone $\mathbf{K}$.
- The necessary capital should be allocated at time zero, with the exchange rules determined by a non-random exchange cone K_0.
- Then the family of all possible initial capital requirements is given by

$$A_0 = \rho_{\mathrm{s}}(X + \mathbf{K}) + (-K_0)\,.$$

- Optimal capital requirements are given by the extremal points from A_0 in the order generated by the cone K_0.
- If A_0 is the whole space, then it is possible to make an infinite capital gain (risk arbitrage).

Risk arbitrage: step by step

- Two currencies: exchange rate π at time one lognormally distributed. No transaction costs.
- Position at time one $X = (0, 0)$.
- Position $(-a, \pi a)$ is reachable from $X = (0, 0)$ at price zero.
- Its risk is $(a, ar(\pi))$.
- So we need a of the first currency and $a\rho(\pi)$ in the second (note that $r(\pi) < 0$).
- If the exchange rate at tme zero is π_0, this costs

$$\pi_0 a + ar(\pi) = a(\pi_0 + r(\pi))$$

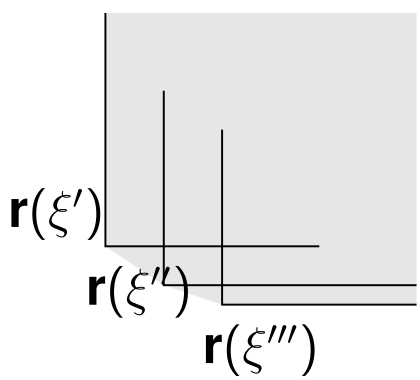
- If π_0 is small enough, then $\pi_0 + r(\pi) < 0$ and we let a grow to release infinite capital.
- The model does not admit financial arbitrage (there exists a martingale measure).

Bounds

- The family of all selections of **X** is immense.
- Exact computation of $\rho_s(\mathbf{X})$ requires multicriterial optimisation algorithms.
- Aim: provide bounds for the selection risk measure.

Upper bound

- Consider any selections $\xi_1, \dots, \xi_N$ of $\mathbf{X}$.
- Determine $\mathbf{r}(\xi_1), \dots, \mathbf{r}(\xi_N)$.
- Take the convex hull of the union of the corresponding upper quadrants.
- This corresponds to a higher risk (subset of $\rho_s(\mathbf{X})$).



Lower bound: univariate example

- A univariate coherent L^p-risk measure r can be represented as

$$r(\xi) = \sup_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(-\zeta\xi)}{\mathbf{E}\zeta},$$

where $\mathcal{Z} \subset L^q(\mathbb{R})$.

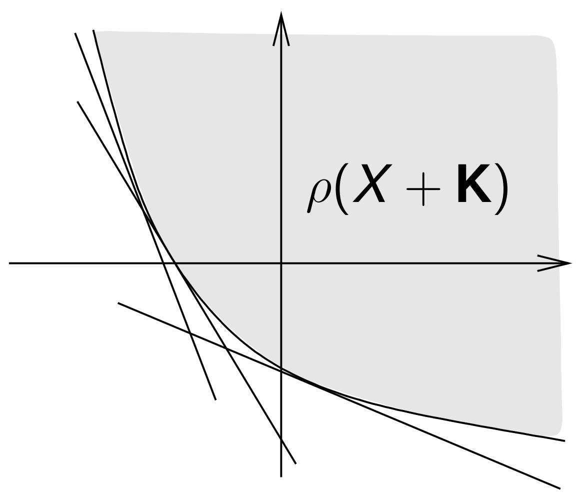
- An alternative expression would be

$$r(\xi) = \bigcap_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}([-\xi, \infty)\zeta)}{\mathbf{E}\zeta} = \bigcap_{\zeta \in \mathcal{Z}} \frac{\mathbf{E}(\check{\mathbf{X}}\zeta)}{\mathbf{E}\zeta},$$

where $\mathbf{X} = (-\infty, \xi]$.

Lower bound

- Intersection of half-spaces:



Lower bound

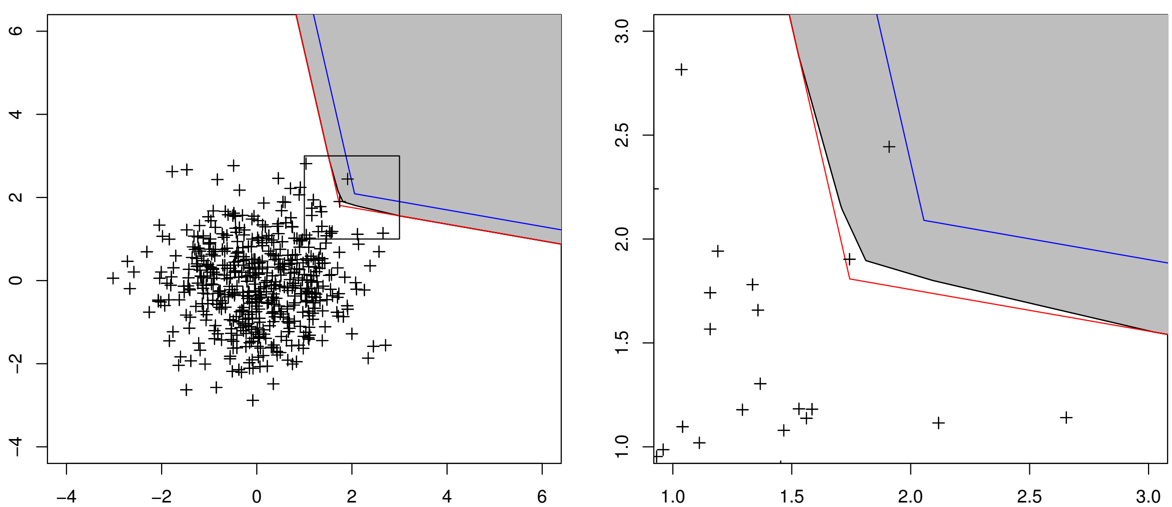
$$\rho_{\mathrm{s}}(\mathbf{X}) \subset \bigcap_{Z \in \mathbf{Z}, u \in \mathbb{R}_+^d} \{x : \mathbf{E}\langle x, uZ\rangle \geq -\mathbf{E} h_{\mathbf{X}}(uZ)\},$$

where

- **Z** is a family of $Z = (\zeta_1, \dots, \zeta_d) \in L^q(\mathbb{R}^d)$ such that ζ_i belongs to a set $\mathcal{Z}_i$ providing the dual representation for r_i, $i = 1, \dots, d$.
- $h_{\mathbf{X}}(\cdot)$ is the support function of **X**.

Normal distribution $\mathbf{X} = X + K$

X is standard normal;
K has $(-5, 1)$ and $(1, -5)$ on its boundary



Dual representation: univariate

- r is a coherent risk measure that is lower semicontinuous, i.e. $r(\xi) \leq \liminf r(\xi_n)$ if $\xi_n \to \xi$ in L^p, $p \in [1, \infty)$.
- For $p = \infty$ one requires the weak star convergence $\xi_n \to \xi$, i.e. $\xi_n \to \xi$ in probability and $|\xi_n| \leq 1$ a.s. for all n.
- Then
$$r(\xi) = \sup_{Z \in \mathcal{Z}} \frac{\mathbf{E}(-XZ)}{\mathbf{E}Z}$$
where $\mathcal{Z}$ is a family of random variables in L^q.

Dual representation

- Replace random set $\mathbf{X}$ with its support function

$$h_{\mathbf{X}}(u) = \sup\{\langle u, x\rangle : \; x \in \mathbf{X}\}\,.$$

- Sums of sets turn into sums of support functions.
- However, $h_{\mathbf{X}}(u)$ may be infinite (even with probability one for all u), e.g. if $\mathbf{X}$ is a half-space.
- Idea: consider $h_{\mathbf{X}}(Y)$, where Y is a selection of the random set (on the unit sphere) being the efficient domain of the support function of $\mathbf{X}$.

Support functions

- $\mathbb{G} = \{u : \; h_{\mathbf{X}}(u) < \infty\}$ is a random cone.
- $\mathbb{G}_1 = \{u \in \mathbb{G} : \; \|u\| = 1\}$ is a random subset of the unit sphere.
- Consider random sets such that $h_{\mathbf{X}}$ is p-integrable and a.s. Lipschitz on $\mathbb{G}_1$ with the p-integrable Lipschitz constant $\|\mathbf{X}\|_{\mathrm{Lip}}$.
- Then we say that $\mathbf{X}$ (or $h_{\mathbf{X}}$) belongs to the space $\mathrm{Lip}^p(\mathbb{G}_1)$.

Theorem

This is the case if and only if $\zeta = \sup_{u \in \mathbb{G}_1} |h_{\mathbf{X}}(u)|$ *is* p*-integrable.*

- For $\mathbf{X} = X + \mathbf{K}$ the condition amounts to the p-integrability of X and $\mathbb{G}$ is the dual cone to $\mathbf{K}$.

Lipschitz space built from support functions

- Consider the space $\mathrm{Lip}^p(\mathbb{G}_1)$ of functions

$$h_{\mathbf{X}}(\eta) : \ (L^0(\mathbb{G}_1), \|\cdot\|_\infty) \mapsto L^p(\mathbb{R})$$

- These maps are Lipschitz, since

$$\mathbf{E}|h_{\mathbf{X}}(\eta) - h_{\mathbf{X}}(\eta')|^p \leq \mathbf{E}\|X\|^p_{\mathrm{Lip}} \|\eta - \eta'\|_\infty \,.$$

- Linear functionals on $\mathrm{Lip}^p(\mathbb{G}_1)$ are given by

$$\mathbf{X} \mapsto \mathbf{E} \sum_{i=1}^{n} h_{\mathbf{X}}(Z_i)$$

where $Z_1, \dots, Z_n \in L^q(\mathbb{G})$.

Linear functionals

- While the weak duals to Lipschitz spaces are not yet known, the weak-star convergence $h_{\mathbf{X}_n}$ to $h_{\mathbf{X}}$ is well understood (Johnson, 1970):
  - pointwise weak-star convergence $h_{\mathbf{X}_n}(\eta) \to h_{\mathbf{X}}(\eta)$ in $L^p(\mathbb{R})$;
  - uniform boundedness of $h_{\mathbf{X}_n}$.

Theorem

$\mathbf{X}_n$ weak-star converges to $\mathbf{X}$ if and only if the Hausdorff distance $\rho_H(\mathbf{X}_n, \mathbf{X}) \to 0$ in L^p if $p \in [1, \infty)$ and in probability together with $\mathbf{X}_n \subset M + \mathbb{G}'$ for a deterministic set M and all n for $p = \infty$.

- In case $\mathbf{X}_n = X_n + K$ for a deterministic exchange cone K corresponds to the usual weak-star convergence $X_n \to X$.

Dual representation

- Fatou property:
  If $\mathbf{X}_n$ weak-star converges to $\mathbf{X}$, then

$$\limsup \rho_{\mathrm{s},0}(\mathbf{X}_n) \subset \rho_{\mathrm{s},0}(\mathbf{X})\,.$$

- Equivalent to the weak-star closedness of the family $\{\mathbf{X} \in \mathrm{Lip}^p(\mathbb{G}_1) :\ \rho_{\mathrm{s},0}(\mathbf{X}) \ni 0\}$.

Theorem

If ρ_{s} has the Fatou property, then

$$\rho_{\mathrm{s}}(\mathbf{X}) = \bigcap_{Z\in\mathbf{Z},u\in\mathbb{R}^d_+} \{x :\ \mathbf{E}\langle x, uZ\rangle \geq -\mathbf{E}h_{\mathbf{X}}(uZ)\}$$

for a certain family $\mathbf{Z} \subset L^q(\mathbb{R}^d)$.

Classical Fatou lemma for selection expectation

- Define $\mathbf{E}_{\mathrm{I}}\mathbf{X} = \{\mathbf{E}\xi : \xi \in L^1(\mathbf{X})\}$ (the Aumann integral, i.e. the expectation without closure).
- Aumann (1965), Schmeidler (1970), etc.
  If $\sup_n \|\mathbf{X}_n\|$ is integrable (random sets in $\mathbb{R}^d$ are uniformly integrably bounded), then

$$\limsup \mathbf{E}_{\mathrm{I}}(\mathbf{X}_n) \subset \mathbf{E}_{\mathrm{I}} \limsup \mathbf{X}_n \,.$$

- Implies the closedness of $\mathbf{E}_{\mathrm{I}}\mathbf{X}$ if we take $\mathbf{X}_n = \mathbf{X}$.
- Generalisations for unbounded random sets are complicated and restrictive (Balder and Hess, 1995).

Fatou property for selection risk measures

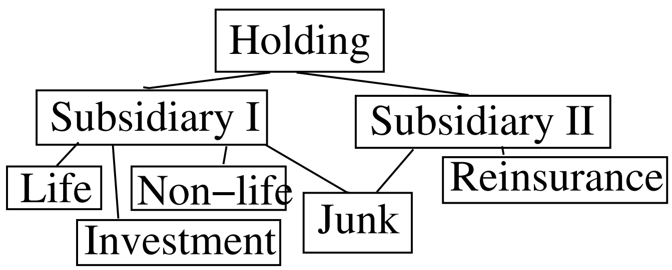
- Fatou property:
  - If $\mathbf{X}_n$ weak-star converges to $\mathbf{X}$, then

$$\limsup \rho_{\mathrm{s},0}(\mathbf{X}_n) \subset \rho_{\mathrm{s},0}(\mathbf{X}) .$$

  holds
- in the L^p-case with $p \in [1, \infty)$;
- in the L^∞-case if $\mathbf{X}$ is quasi-bounded;
- in the L^∞-case if $\mathbf{X} = X + K$ is for deterministic cone K.

In all these cases $\rho_{\mathrm{s},0}(\mathbf{X})$ is closed.

Financial networks



- IntraGroup Transfers (in order to help distressed members of the group)

Allianz Group

Allianz

Allianz Group Structure

This overview is simplified. It focuses on major operating entities and does not contain all entities of Allianz Group. It does not show whether a shareholding is direct or indirect. Indications show status as of December 31, 2013.

Allianz SE

H4 – Operations

- Allianz Global Assistance S.A.S.[1] – France, Paris
- Allianz Worldwide Care Ltd.[1] – Ireland, Dublin

H5 – Insurance Western Europe & Southern Europe

- Allianz S.p.A. – Italy, Trieste
- Allianz Vie S.A. – France, Paris
- Allianz Belgium S.A. – Belgium, Brussels
- Allianz IARD S.A. – France, Paris
- Allianz Nederland Groep N.V. – Netherlands, Rotterdam
- Allianz Hellas Insurance Company S.A. – Greece, Athens
- Allianz Hayat ve Emeklilik A.S. – Turkey, Istanbul
- Allianz Sigorta A.S. – Turkey, Istanbul
- Yapi Kredi Sigorta A.S. – Turkey, Istanbul
- Allianz Yasam ve Emeklilik A.S. – Turkey, Istanbul

H6 – Insurance USA

- Allianz Life Insurance Company of North America – United States, Minneapolis (MN)
- Fireman‘s Fund Insurance Company Corp. – United States, Novato (CA)

H7 – Insurance German Speaking Countries

- Allianz Versicherungs-AG – Germany, Munich
- Allianz Lebens-versicherungs-AG – Germany. Stuttgart
- Allianz Private Kranken-versicherungs-AG – Germany, Munich
- Allianz Beratungs- und Vertriebs-AG – Germany, Munich
- Allianz Elementar Versicherungs-AG – Austria, Vienna
- Allianz Elementar Lebens-versicherungs- AG – Austria, Vienna
- Allianz Suisse Versicherungs-Gesellschaft AG – Switzerland, Zurich
- Allianz Suisse Lebensversicherungs-Gesellschaft AG – Switzerland, Zurich
- Oldenburgische Landesbank AG – Germany, Oldenburg

H8 – Insurance Growth Markets

- Allianz Hungária Biztosító Zrt. – Hungary, Budapest
- Allianz-Slovenská poist'ovna a.s. – Slovakia, Bratislava
- Allianz pojistovna a.s. – Czech Republic, Prague
- TU Allianz Polska S.A. – Poland, Warsaw
- OJSC Insurance Company Allianz – Russia, Moscow
- Allianz Other CEE (Croatia, Bulgaria, Romania)
- Allianz Life Insurance Co. Ltd. – Korea, Seoul
- Allianz Taiwan Life Insurance Co. Ltd. – Taiwan, Taipei
- Allianz Malaysia Berhad p.l.c. – Malaysia, Kuala Lumpur
- Allianz MENA (Egypt, Lebanon)
- Allianz Other Asia-Pacific (Indonesia, Thailand)

H9 – Asset Management Worldwide

- Allianz Global Investors Europe GmbH – Germany, Frankfurt am Main
- PIMCO Deutschland GmbH – Germany, Munich
- Allianz Global Investors France S.A. – France, Paris
- Allianz Global Investors Asia Pacific Group
- Pacific Investment Management Company LLC – United States, Dover (DE)
- Allianz Asset Management of America – United States, Dover (DE)

H10 – Global Insurance Lines & Anglo Markets

- Allianz Insurance plc – United Kingdom, Guildford
- Allianz p.l.c. – Ireland, Dublin
- Allianz Australia Limited – Australia, Sydney
- Allianz Global Corporate & Specialty SE – Germany, Munich
- Allianz Global Risks US Insurance Company Corp. – United States, Burbank (CA)
- Euler Hermes S.A. – France, Paris

H11 – Insurance Iberia & Latin America

- Allianz compañía de Seguros y Reaseguros S.A. – Spain, Barcelona
- Allianz Popular S.L. – Spain, Madrid
- Companhia de Seguros Allianz Portugal S.A. – Portugal, Lisbon
- Allianz Latin America (Brazil, Argentina Colombia, Mexico)

The functional divisions

H1 – Chairman of the Board

H2 – Finance, Controlling, Risk

H3 – Investments

do not have major operating entities in their area of responsibilities and are therefore not shown in the overview.

1) Starting January 1, 2014: AWP Group, including Allianz Global Assistance and Allianz Worldwide Care

Stand-alone regulation

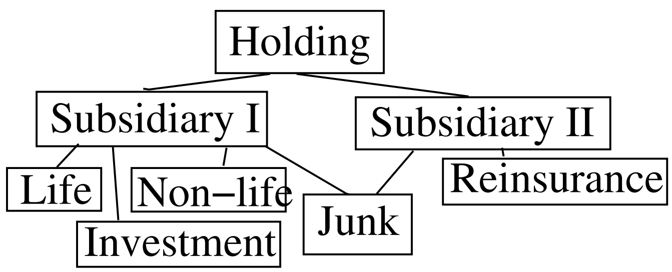
- Consider each agent separetely and assess his liabilities and gains (balance sheets).
- Suggest a probability model for his terminal position C_i.
- Calculate the risk $r(C_i)$ using the chosen risk measure.
  - In the EU mostly used Value-at-Risk: it is not subadditive and is not sensitive for high losses.
  - In Switzerland: Expected Shortfall (more conservative and secure).
- Request each agent to reserve capital $r(C_i)$ (or allow to release capital if $r(C_i)$ is negative).

Systemic risk

- In a network possible default of an agent increases the pressure on the whole network.
- Possible influences of losses between the agents should be taken into account.
- This leads to an increase of the required capital reserves.

Bulding groups

- If the agents form a group, some may voluntarily (or may be forced by the holding) compensate for losses of other group members.
- This may and should lead to a decrease of capital reserves.



Example: Financial network, two agents

- Two agents A and B are assessed by a regulator.
- The regulator evaluates their individual exposure to risks and requests that they set aside (and freeze) necessary capital reserves.
- The agents want to minimise these reserves and conclude an agreement that at the terminal time (and under certain conditions) one of them would help to offset the deficit of another one.
- The family of allowed transactions is a random closed set in the plane.
- Despite the network is not geometric, it is described by a geometric object.

Assumptions

- d agents operate with the same currency.
- Their terminal positions are $C_1, \ldots, C_d$ (assume $d = 2$).
- All risks are assessed using the same coherent risk measure r (say Expected Shortfall).

Unrestricted transfers

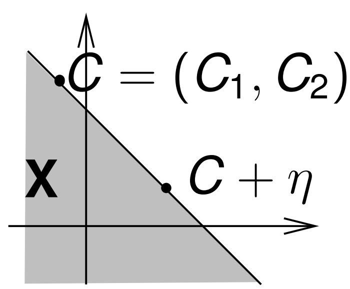
- The regulator allows all transfers.
- Moving from $C = (C_1, \ldots, C_d)$ to

$$C + \eta = (C_1 + \eta_1, \ldots, C_d + \eta_d)$$

  with

$$\eta_1 + \cdots + \eta_d \leq 0.$$

- The resulting attainable positions build a half-space.



Theorem: Jouini, Schachermayer, Touzi, Filipovic, Kupper

- The total required capital for all agents is

$$r(C_1 + \cdots + C_d)$$

It is less than the capital in the stand-alone approach

$$r(C_1 + \cdots + C_d) \leq \sum r(C_i)$$

- There exist "best" transfers:

$$r(C_1 + \cdots + C_d) = \inf_{\eta:\, \sum \eta_i \leq 0} \sum r(C_i + \eta_i),$$

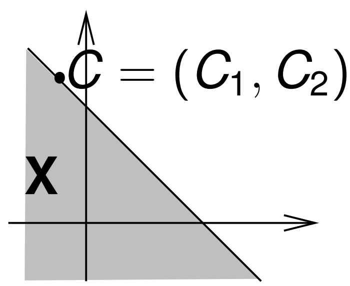
so the infimum is attained at $\eta = \eta^*$, the transfers do not worsen the risk assessment of each of agent:

$$r(C_i + \eta_i^*) \leq r(C_i),$$

and $C_i + \eta_i = f(\sum C_i)$ are all monotonic functions of the total position.

Unrestricted transfers

- The result can be alternatively derived using the closedness of the risk measure of the random set **X**.

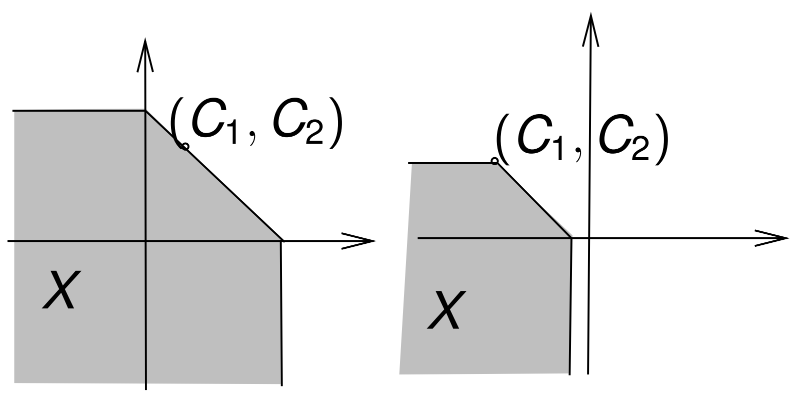


Imposing restrictions

- The optimal transfers may lead to a bankruptcy of some agents.
- The capital available for transfers normally is not available as cash: the agents have to take a loan using their assets as a deposit.
- The interest on such loan depends on the balance sheet of the agent (fungibility).

Example: no transfers that lead to bankruptcy

- The terminal capital is (C_1, C_2).
- Transfers from a solvent company to another one are allowed up to the available positive capital.
- No fungibility difficulties.
- Disposal of assets is allowed (e.g. as dividends to shareholders).



- $\mathbf{X} = \mathbf{X}(C)$ is a random set of attainable positions for group members.
- Determine

$$\inf\{\sum a_i : \ 0 \in \rho_s(\mathbf{X}(C + a))\}$$

This is the minimal total capital that should be reserved in order to make $\mathbf{X}(C + a)$ acceptable as a random set.

Theorem

The infimum is achieved.

- How to ensure that none of the agents needs to reserve more than it would do in case of a stand-alone approach?
- In which case the optimal transfers are realised as a monotonic function of $\mathbf{X}$?

Summary

- In many applications one deals with non-stationary random sets.
- Compact random sets or possibly unbounded random sets or sometimes even non-convex (indivisible assets), possibly high-dimensional.
- Selections and support functions form the central tool to solve the corresponding problems.
- Models, inference tools, and efficient computation algorithms are needed.

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- Works in progress ...