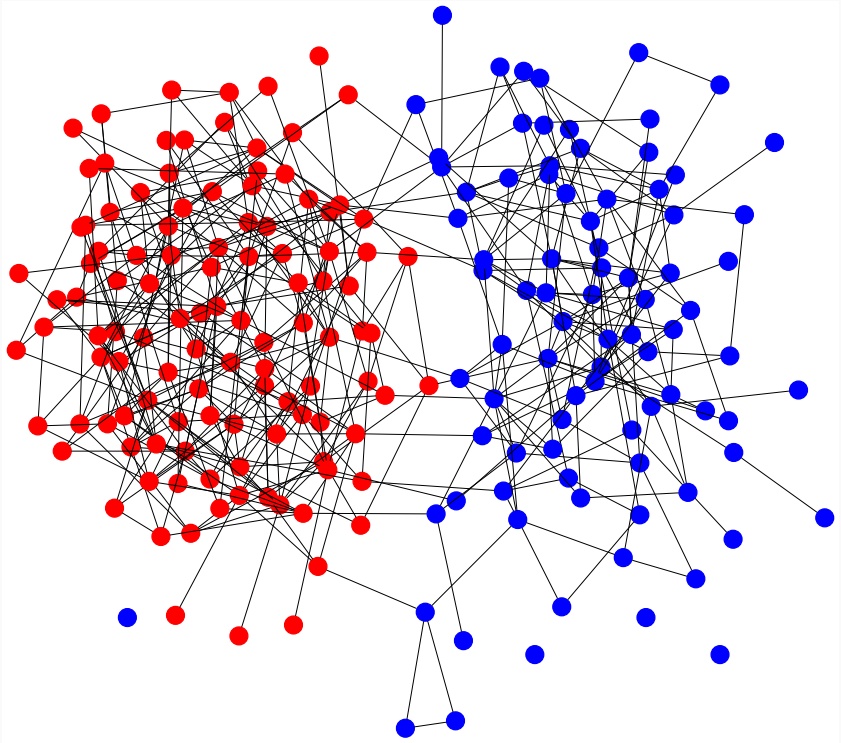
The condensation threshold in stochastic block models

Joe Neeman (with Jess Banks, Cris Moore, Praneeth Netrapalli)

Austin, May 9, 2016

Stochastic block model $\mathcal{G}(n, k, a, b)$

1. n nodes, k colors, about n/k nodes of each color
2. connect u to v with probability $\begin{cases} \frac{a}{n} & \text{if the same color} \\ \frac{b}{n} & \text{if different colors} \end{cases}$



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Definition

$(G_n, \sigma_n) \sim \mathcal{G}(n, k, a, b)$ is *detectable* if there exists $\epsilon > 0$ and maps $A_n : \{\text{graphs}\} \to \{\text{labellings}\}$ such that

$$\liminf_{n \to \infty} \Pr(\mathrm{O}_{\mathrm{lap}}(\sigma_n, A_n(G_n)) > \epsilon) > \epsilon.$$

Otherwise it is *undetectable*.

Problem II: distinguishing

Given the (uncolored) graph, did it come from $\mathcal{G}(n, k, a, b)$ or $\mathcal{G}(n, \frac{d}{n})$, where $d = \frac{a+(k-1)b}{k}$?

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Sequences $\mathbb{P}_n$ and $\mathbb{Q}_n$ of probability measures are

- *contiguous* if $\mathbb{P}_n(A_n) \to 0$ iff $\mathbb{Q}_n(A_n) \to 0$
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Say that $\mathcal{G}(n, k, a, b)$ is

- *distinguishable* if it is orthogonal to $\mathcal{G}(n, \frac{d}{n})$
- *indistinguishable* if it is contiguous with $\mathcal{G}(n, \frac{d}{n})$

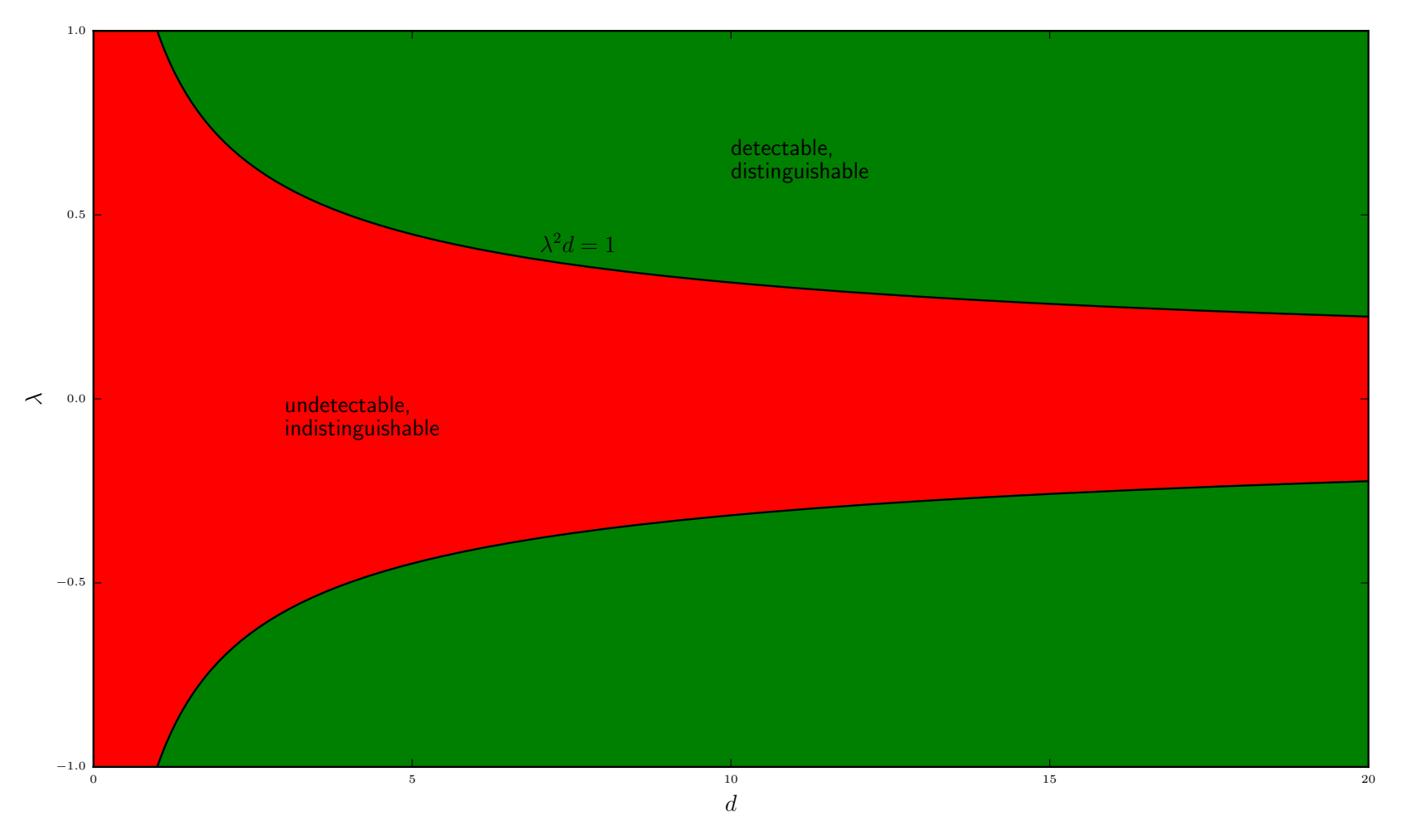
Better parametrization

- $\frac{a}{n}$ = within-block edge probability
- $\frac{b}{n}$ = between-block edge probability
- k = number of blocks

$$d = \frac{a + (k-1)b}{k}$$
$$\lambda = \frac{a - b}{a + (k-1)b}$$

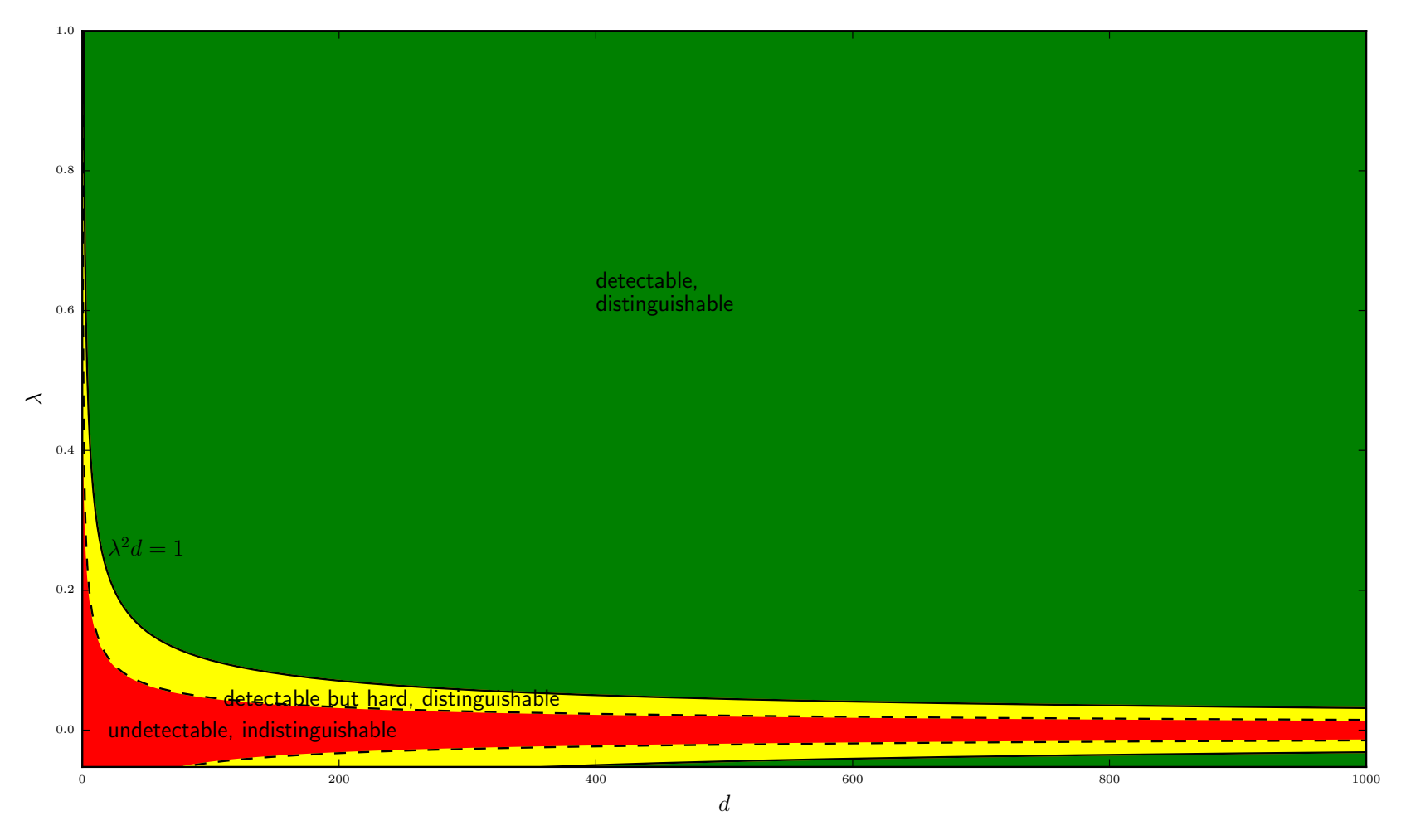
Note $\lambda \in \left[-\frac{1}{k-1}, 1 \right]$.

Phase diagram for $k = 2$



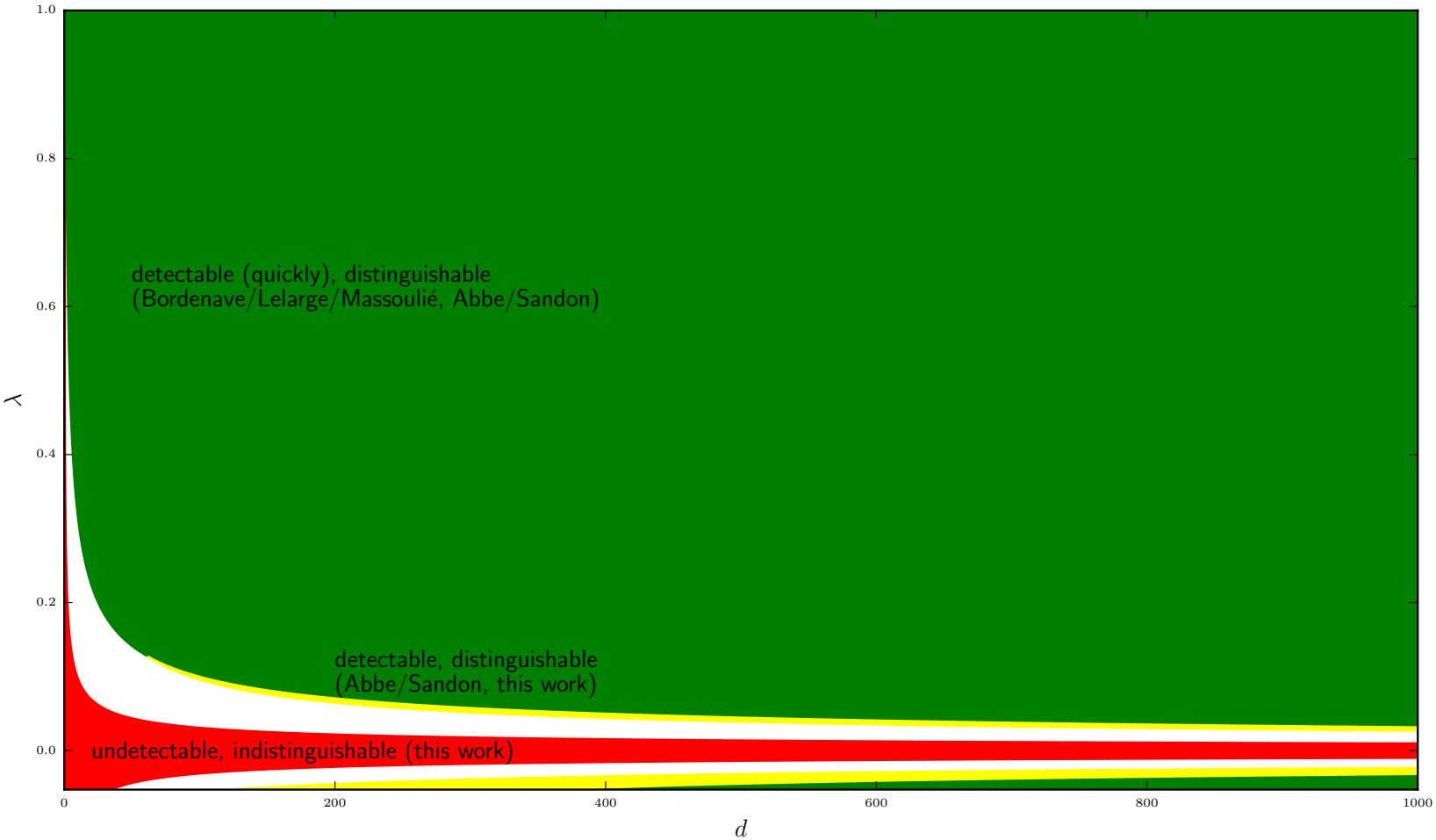
(Mossel/N/Sly, Massoulié)

Conjectured phase diagram for $k = 20$



(Decelle, Krzakala, Moore, Zdeborova)

What we know for $k = 20$



Theorem (Banks/Moore/N/Netrapalli)

$$d^+ = \frac{2k \log k}{(1 + (k-1)\lambda) \log(1 + (k-1)\lambda) + (k-1)(1-\lambda) \log(1-\lambda)}$$

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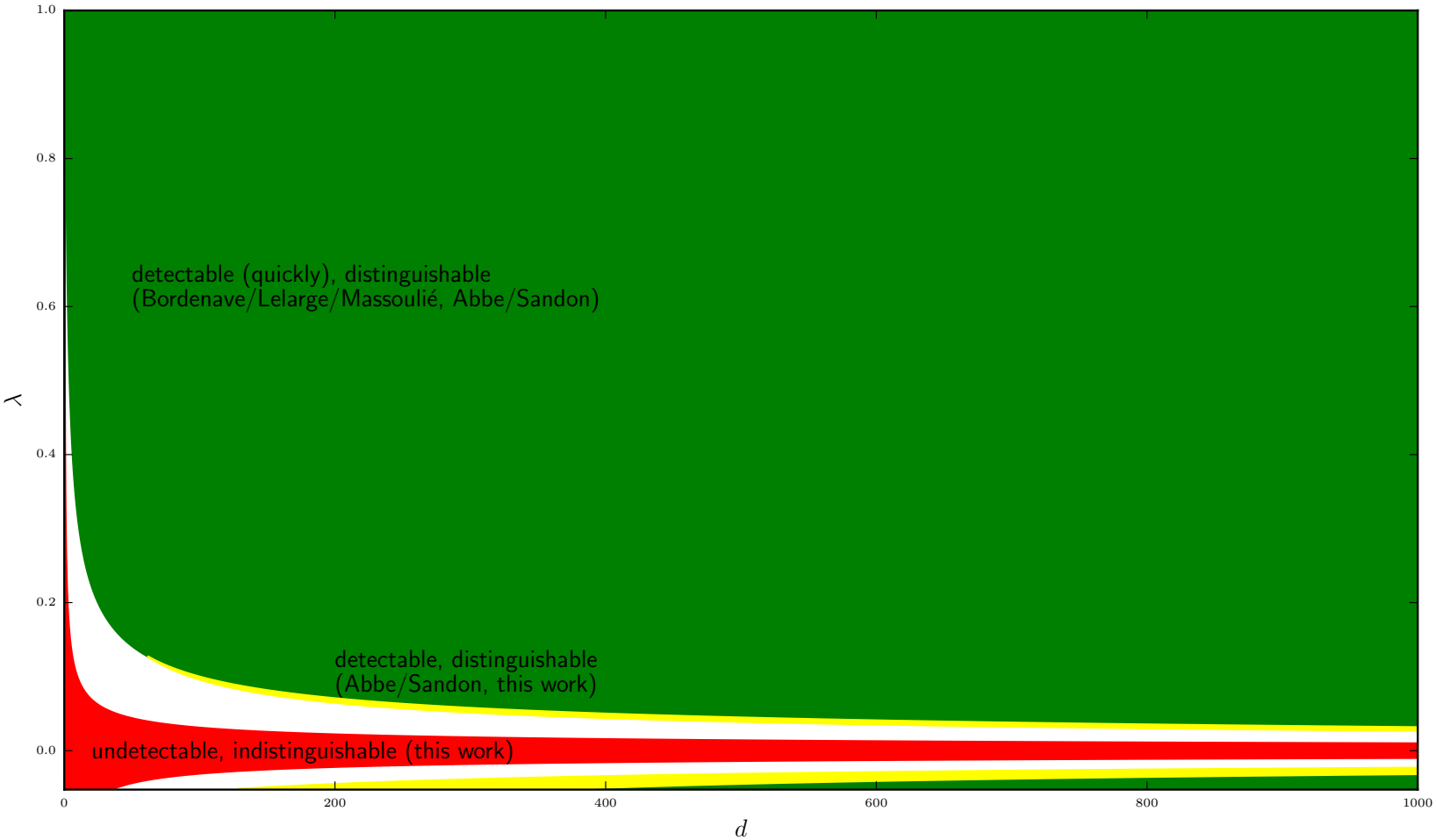
$$\lim_{k\to\infty} \frac{d^+}{d^-} = \frac{\mu^2}{(1+\mu)\log(1+\mu) - \mu} \text{ where } \mu = \frac{a-b}{d}.$$

If $\mu \approx \pm 1$ and $\lim_{k\to\infty} \frac{d^+}{d^-} \approx 1$ (planted coloring / giant)

The proofs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Detecting/distinguishing inefficiently

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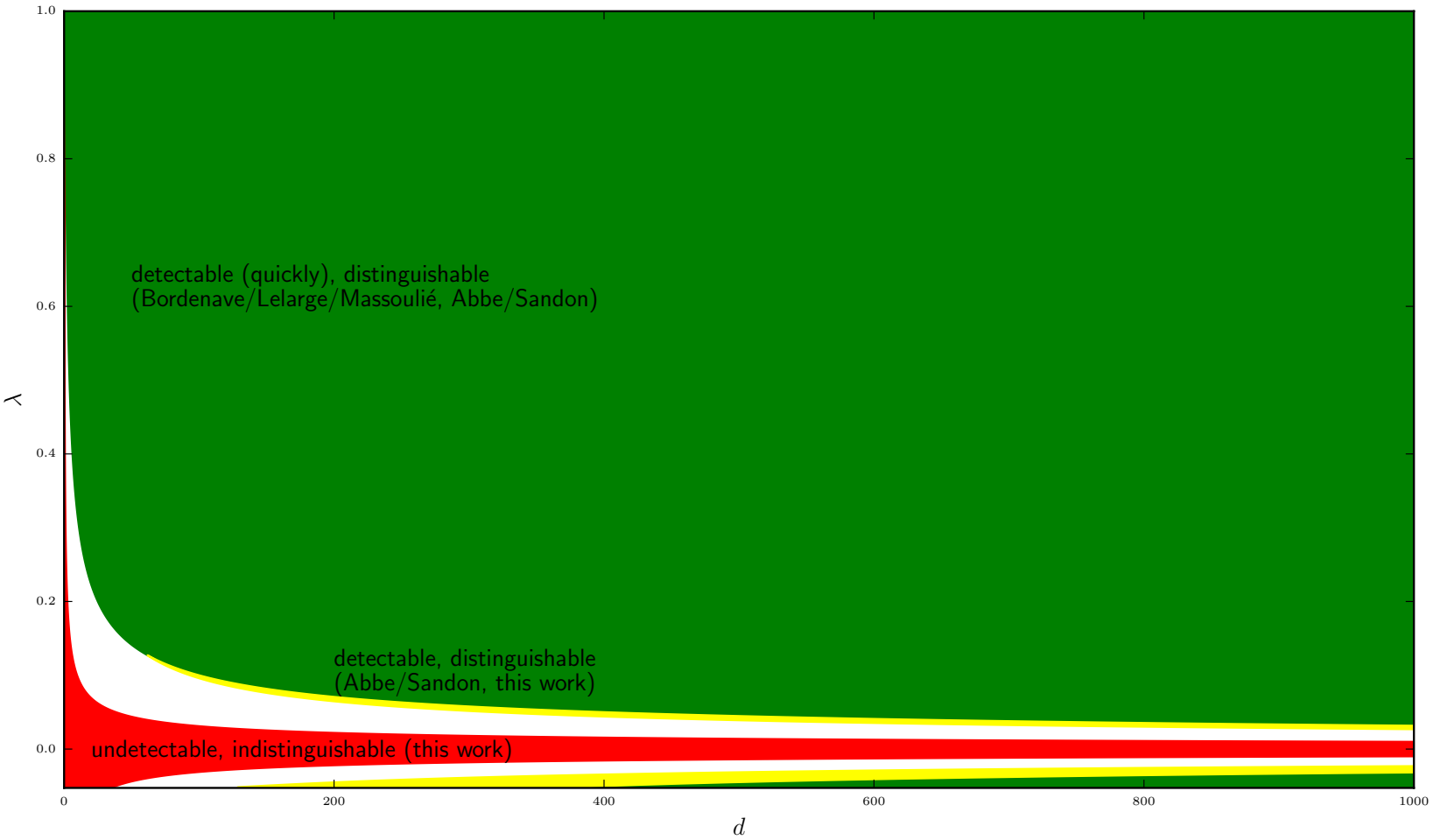
Abbe/Sandon improved this for small d by taking the giant component and pruning trees.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Indistinguishability

If $\mathbb{E}_{\mathbb{Q}_n}\left(\frac{d\mathbb{P}_n}{d\mathbb{Q}_n}\right)^2 \to C < \infty$ then $\mathbb{Q}_n(A_n) \to 0 \Rightarrow \mathbb{P}_n(A_n) \to 0$.

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Under $\mathbb{Q}_n$, the events $(u, v) \in E$ are all independent, so can compute:

$$\mathbb{E}_{\mathbb{Q}_n}\left(\frac{d\mathbb{P}_n}{d\mathbb{Q}_n}\right)^2 = C(1 + o(1))\mathbb{E}\exp(X^T B X),$$

where X is a multinomial vector of length k^2.

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Achlioptas-Naor: sufficient condition for the maximum to be at $x = \mathbb{E}X$. (They were studying planted colorings.)

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Main thing to check: convergence of second moment.

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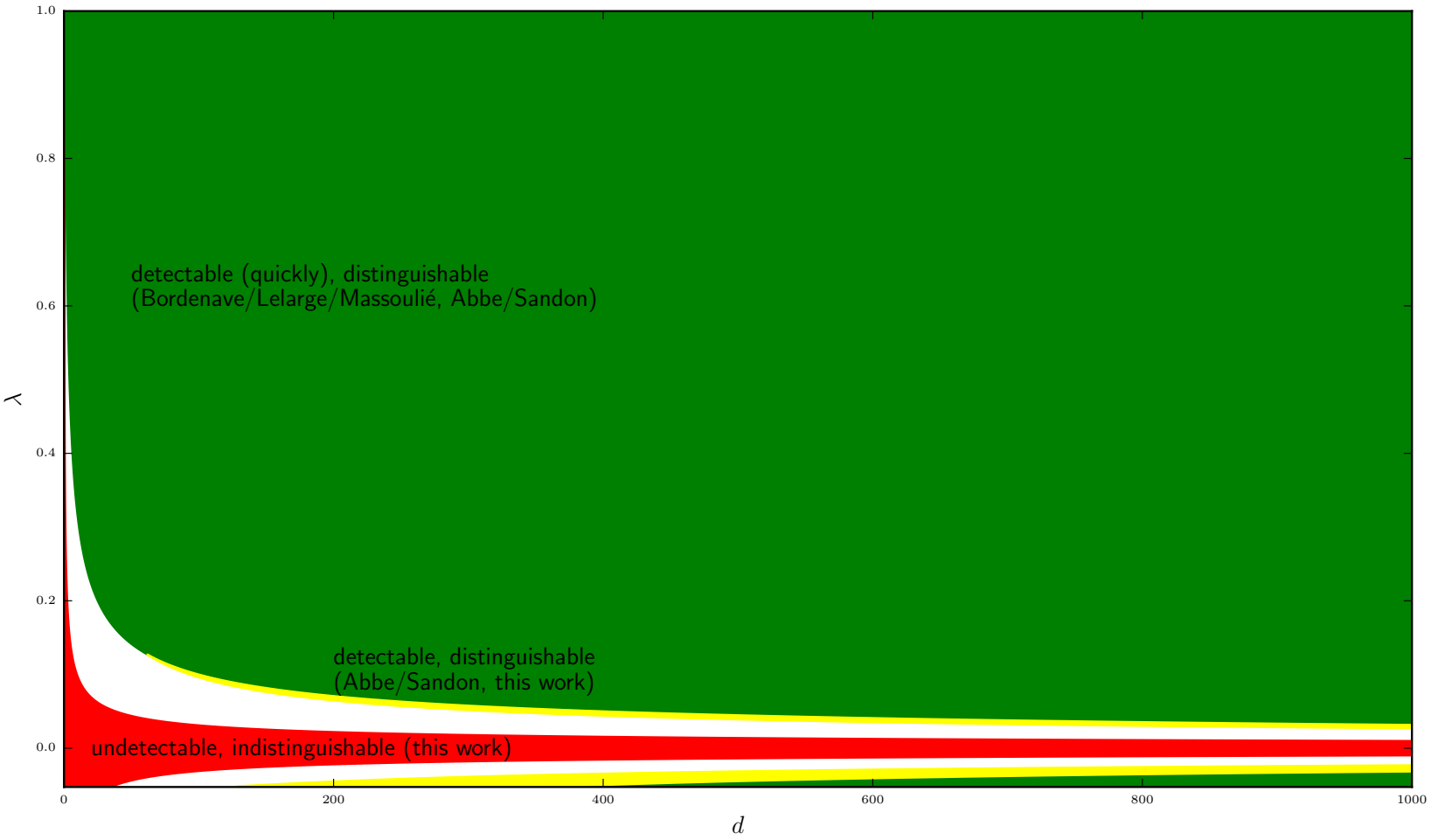
Similar to previous second moment computation.

Summary

Indistinguishability and undetectability follow from an explicit second moment calculation. Use Achlioptas-Naor to estimate the set of parameters where the second moment is finite.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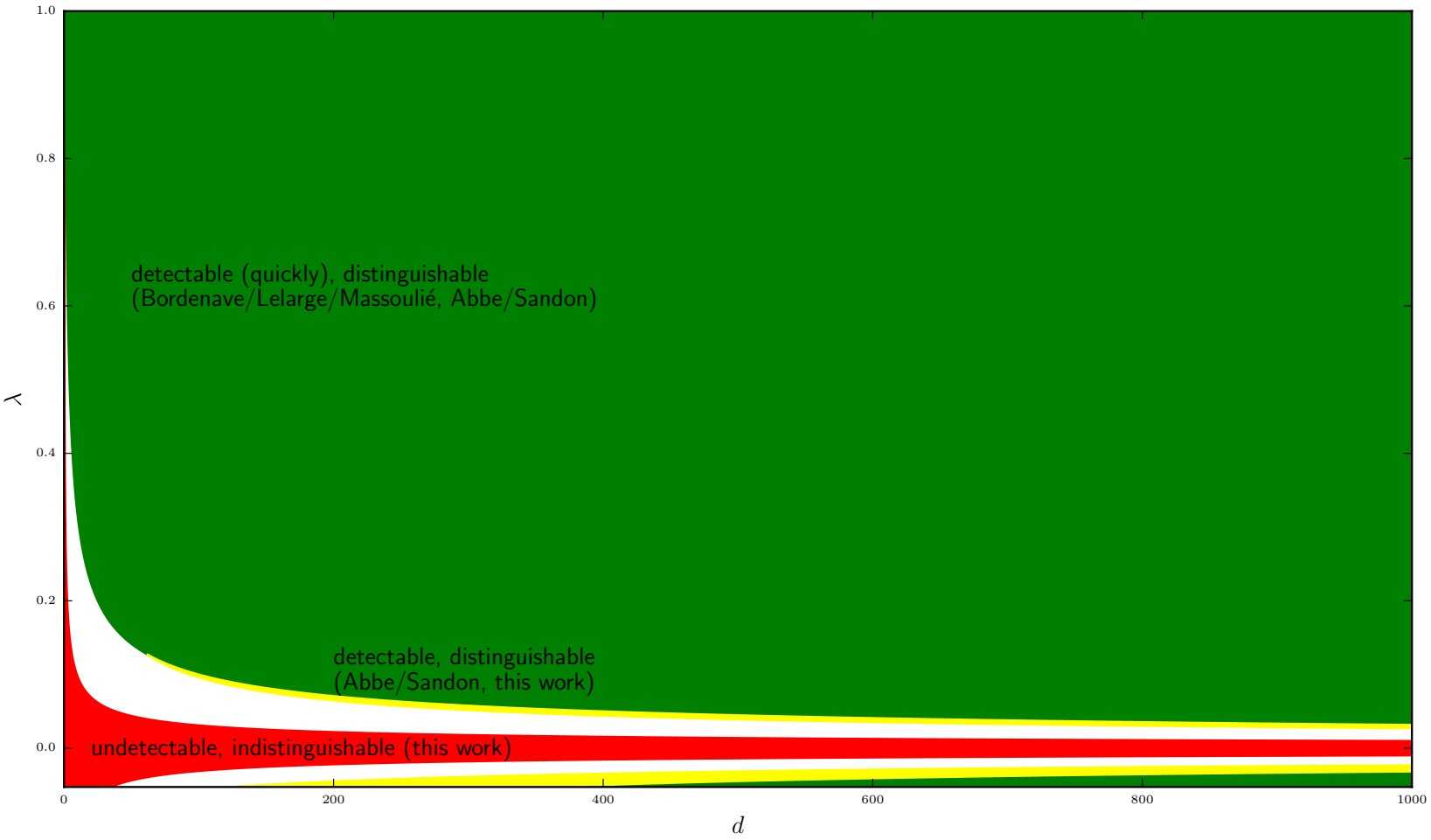
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