Quantum Scattering for Potentials Independent of $|x|$: Asymptotic Completeness for High and Low Energies

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Abstract

Let $V_1 : S^{n-1} \to \mathbb{R}$ be a Morse function and define $V_0(x) = V_1(x/|x|)$. We consider the scattering theory of the Hamiltonian $H = -\frac{1}{2}\Delta + V(x)$ in $L^2(\mathbb{R}^n)$, $n \geq 2$, where V is a short-range perturbation of V_0. We introduce two types of wave operators for channels corresponding to local minima of V_1 and prove completeness of these wave operators in the appropriate energy ranges.

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Contents

1 Introduction

Consider the *classical* Hamiltonian on $(\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n$ given by

$$H(x,\xi) = \frac{1}{2}(\xi - A(x))^2 + V(x),$$

where $A, V \in C^\infty(\mathbb{R}^n \setminus \{0\})$ are *homogeneous of degree zero.* For a scattering orbit (defined by $r(t) = |x(t)| \to \infty$ with $t \to +\infty$), introduce the variables

$$\omega = \frac{x}{r},\ \eta = r\dot{\omega},\ h = H(x,\xi),$$

and a new time τ:

$$\frac{d\tau}{dt} = \frac{1}{r}.$$

For t large we find

$$\frac{d}{d\tau}\begin{pmatrix}\omega \\ \eta\end{pmatrix} = f_h(\omega,\eta). \tag{1.1}$$

This is an *autonomous, non-Hamiltonian* system in a $2(n-1)$ dimensional phase space. The fact that (1.1) is autonomous stems from the homogeneity of V and A. Since the system is non-Hamiltonian, there is a variety of phenomena which can and do occur but which are prohibited in Hamiltonian dynamics. For example if $n = 2$ and $V = 0$ and if the magnetic field $B = \operatorname{curl} A$ has constant sign, the system (1.1) has a globally attracting periodic orbit at high energy. Interestingly, this can also be proved in quantum mechanics [CHS]. Thus from the geometric and dynamical systems point of view, such quantum systems have a rich structure.

In this paper we set A=0 and consider the operator

$$H = -\frac{1}{2}\Delta + V(x)$$

in $L^2(\mathbb{R}^n)$, $n \geq 2$, where in this introduction we assume for simplicity that V is smooth, real, and homogeneous of degree zero outside the open unit ball. In addition, we assume here that $V|S^{n-1}$ has a finite number of non-degenerate critical points, $C_r = \{\omega_1, \ldots, \omega_N\}$. For $e \in C_r$, let P_e be the orthogonal projection onto

$$\left\{\psi \in L^2(\mathbb{R}^n) :\ \lim_{t\to\infty}\left\|\left(e - \frac{x}{|x|}\right)e^{-itH}\psi\right\| = 0\right\}.$$

It follows immediately that

$$P_{\omega_i}P_{\omega_j} = \delta_{ij}P_{\omega_i}, \quad [P_{\omega_i}, H] = 0.$$

Combining results of [He] and [ACH], it follows that

$$\sum_j P_{\omega_j} = P_{\text{cont}}(H),$$

where $P_{\text{cont}}(H)$ is the orthogonal projection onto the continuous spectral subspace of H. In [HS1], it is shown that if $V|S^{n-1}$ has a local maximum at ω_j, then $P_{\omega_j} = 0$. We believe that unless $V|S^{n-1}$ has a local minimum at ω_j, then $P_{\omega_j} = 0$. We proved this under an additional technical assumption in [HS1]; another proof avoiding this assumption is in progress, [HS2].

It is the purpose of this paper to examine more carefully the asymptotic behavior of $e^{-itH}P_e\psi$ where e is the location of a local minimum of $V|S^{n-1}$. We are able to accomplish this to a satisfactory degree in two regions of the continuous spectrum of $H|(\text{Range } P_e)$. These energy regions are given with reference to the corresponding classical system (1.1). The point $(e, 0)$ is a stable fixed point of this system but the character of this fixed point changes with the energy E: Below a certain energy E_0 the fixed point is a spiral sink. All eigenvalues of the linearized system have negative real part and nonzero imaginary part. We have

$$\left|\frac{x(t)}{|x(t)|} - e\right| = \mathcal{O}(t^{-\frac{1}{2}}),$$

as $t \to \infty$. Above a certain energy E_2, the fixed point is a nodal sink. All eigenvalues of the linearized system are negative. There is in general an intermediate region, $E_0 < E < E_2$, where some of the eigenvalues are real and negative while the remainder have negative real part and nonzero imaginary part. In the region $E_0 \leq E$ we have

$$\left|\frac{x(t)}{|x(t)|} - e\right| = \mathcal{O}(t^{-\mu(E)}),$$

with $\mu(E)$ monotonically decreasing from $\frac{1}{2}$ to zero as $E \to \infty$. We single out the energy $E_1 > E_0$ below which we have $\mu(E) > \frac{1}{3}$. Energies below E_1 will be called "low" and those above E_2 will be called "high".

We obtain a detailed enough description of the asymptotic behavior of $e^{-itH}\psi$ in the high and low energy regions to prove asymptotic completeness results in these regimes. Let $p = -i\nabla$ and suppose for definiteness that $e = e_1 = (1, 0, \ldots, 0)$. We normalize V so that $V(e_1) = 0$. The result in the low energy region is easiest to describe: Define a comparison dynamics by giving the time-dependent Hamiltonian

$$H_0(t) = \frac{1}{2}p^2 + \frac{1}{2}\langle x'_\perp, V^{(2)}(e_1)x'_\perp\rangle/(tp_1)^2.$$

Here $V^{(2)}(e_1)$ is the Hessian of $V|S^{n-1}$ at e_1 and $x'_\perp = (0, x_2, \ldots, x_n)$. We put $x_\perp = (x_2, \ldots, x_n)$ and $p_\perp = (p_2, \ldots, p_n)$. The second term in the Hamiltonian $H_0(t)$ is obtained by expanding $V\left(1, \frac{x_\perp}{x_1}\right)$ $(= V(x)$ for x_1 large and positive) in a power series in $\frac{x_\perp}{x_1}$, keeping only the first non-vanishing term which is quadratic in $\frac{x_\perp}{x_1}$, and then replacing x_1 by tp_1. Notice that p_1 commutes with $H_0(t)$ and that after fixing p_1 at ξ_1, $H_0(t)$ is quadratic in $p_\perp$ and $x_\perp$. A simple transformation shows that the propagator $U_0(t)$ satisfying $i\partial_t U_0(t) = H_0(t)U_0(t)$ can be related to that of a harmonic oscillator if

$\frac{\xi_1^2}{2} < E_0$. If $\frac{\xi_1^2}{2} > E_0$ the term in the resulting Hamiltonian quadratic in $x_\perp$ is no longer positive. In fact, we have explicitly

$$U_0(t) = S_{t^{-\frac{1}{2}}} e^{\frac{i|x_\perp|^2}{4}} e^{-\frac{itp_1^2}{2}} e^{-i(\ln t)H_2} \hat{U}_0,$$

where $\hat{U}_0$ is a constant unitary operator at our disposal,

$$\hat{U}_0 = e^{-\frac{i|x_\perp|^2}{4}} e^{\frac{ip_1^2}{2}} U_0(1),$$

$S_{t^{-\frac{1}{2}}}$ is a scale transformation

$$S_{t^{-\frac{1}{2}}} f(x_1, x_\perp) = t^{-\frac{n-1}{4}} f\left(x_1, \frac{x_\perp}{\sqrt{t}}\right),$$

and

$$H_2 = \frac{1}{2}p_\perp^2 + \frac{1}{2}\left\langle x'_\perp, \left(p_1^{-2} V^{(2)}(e_1) - \frac{1}{4}I\right) x'_\perp \right\rangle.$$

For simplicity we take $U_0(1) = I$. Asymptotic completeness takes the following form in the energy range $0 < E < E_1$. (A more general result is given in Theorem 3.1.)

Theorem 1.1 *Let χ be the indicator function of $\left\{\xi_1 : \xi_1 > 0, \frac{\xi_1^2}{2} < E_1\right\}$. Define*

$$\mathcal{H}_1 = \chi(p_1) L^2(\mathbb{R}^n), \ \mathcal{H}_2 = P_{e_1} E_H((0, E_1)) L^2(\mathbb{R}^n),$$

where $E_H(F)$ is the spectral projection for H in the Borel set $F \subset \mathbb{R}$. Then the strong limit

$$\Omega = \lim_{t\to\infty} e^{itH} U_0(t)$$

exists on $\mathcal{H}_1$ and defines a unitary operator

$$\Omega : \mathcal{H}_1 \xrightarrow{\text{onto}} \mathcal{H}_2.$$

The simple approximation used to obtain $H_0(t)$ from H has much in common with Dollard's idea for constructing wave operators to describe Coulomb scattering. The wave operators we construct in the high energy regime are similar to those introduced by Yafaev [Yaf] to describe long-range scattering (see [Yaf] for existence and [DG1] for completeness). For this purpose we need a suitable solution $S(t, x)$ to the Hamilton-Jacobi equation

$$-\partial_t S(t, x) = \frac{1}{2}|\nabla_x S(t, x)|^2 + V(x). \tag{1.2}$$

We are able to construct such a solution in a region of the form

$$\left\{(t, x) : \frac{x_1}{t} > \sqrt{2E_2}, \ \frac{x_1}{t} \notin \mathcal{R}, \ \left|\frac{x_\perp}{t}\right| \text{ small}, \ t \text{ large}\right\},$$

where $\mathcal{R}$ is a discrete set of "resonances" (see Section 2 for a precise definition of S and its domain). Our propagator $\tilde{U}_0(t)$ solves a first-order PDE:

$$i\partial_t \tilde{U}_0(t) = \left(H - \frac{1}{2}(p - \nabla_x S(t,x))^2\right) \tilde{U}_0(t),$$

and, in fact, is given explicitly by the formula

$$(\tilde{U}_0(t)f)(x) = e^{iS(t,x)}(J(t,x))^{\frac{1}{2}} f(k(t,x), w(t,x)),$$

where $k(t,x)$ (> 0) is related to the energy of the classical orbit $x(s)$ satisfying $\frac{dx(s)}{ds} = \nabla_x S(s, x(s))$, which goes through the point x at time t. Explicitly

$$\frac{(k(t,x))^2}{2} = \frac{1}{2}|\nabla_x S(t,x)|^2 + V(x).$$

The quantity $w(t,x) \in \mathbb{R}^{n-1}$ is an observable associated with the asymptotics of this orbit which will be described in Section 2. (For a somewhat different interpretation see the remark after Theorem 3.2.) $J(t,x)$ is the Jacobian which makes $\tilde{U}_0(t)$ isometric. The formula above involving $\tilde{U}_0(t)$ must be properly interpreted because the domain of definition of $S(t,x)$ is somewhat complicated.

Asymptotic completeness in the high energy regime takes the form (a more general result is given in Theorem 3.2):

Theorem 1.2 *Let*

$$\tilde{\mathcal{H}}_1 = L^2((\sqrt{2E_2}, \infty) \times \mathbb{R}^{n-1}) \text{ and } \tilde{\mathcal{H}}_2 = P_{e_1} E_H((E_2, \infty)) L^2(\mathbb{R}^n).$$

For each $f \in C_0^\infty((\sqrt{2E_2}, \infty) \setminus \mathcal{R} \times \mathbb{R}^{n-1})$ *the limit*

$$\tilde{\Omega} f = \lim_{t\to\infty} e^{itH} \tilde{U}_0(t) f$$

exists and extends by continuity to a unitary operator

$$\tilde{\Omega} : \tilde{\mathcal{H}}_1 \xrightarrow{\text{onto}} \tilde{\mathcal{H}}_2.$$

Remarks. (1) One might expect that a wave operator similar to that used in "long-range" scattering theory would be possible to construct. However, if

$$\lim_{t\to\infty} e^{itH} e^{-iW(t,p)} f = \psi,$$

exists and lies in Range P_{e_1} for some real function $W(t,\xi)$, then according to Lemma 3.5 if $\epsilon > 0$ and χ_ϵ is the indicator function of $\{\xi : |\xi_\perp| > \epsilon\}$,

$$\lim_{t\to\infty} \chi_\epsilon(p) e^{-itH} \psi = 0,$$

which implies $\chi_\epsilon(p) f = 0$. Thus $f = 0$.

(2) One might expect that the resonances for our solution S to (1.2) are not exceptional points at all so that, more precisely, S may be extended to a smooth solution across those points. However this is not the case, in fact the occurrence of resonances is an intrinsic property of the Hamilton-Jacobi equation (and hence not a deficiency of our method of construction of a solution which is based on the Sternberg linearization scheme): For a generic homogeneous of degree zero potential resonances occur for any solution to (1.2) with a certain homogeneity property and a relevant second order expansion, see (2.21) and (5.60). Presumably they accumulate at infinity.

In the next section some details will be given about the structure of classical orbits which have the property that $t^{-1}x_\perp(t)$ and $p_\perp(t) \to 0$. In particular, the nature of the different energy regimes will be described more fully. It follows easily from this analysis that if $V^{(2)}(e_1)$ is a multiple of the identity, then, in fact, $E_0 = E_2$ so that we have a complete description over the full energy range for scattering into direction e_1 (this is the case for $n = 2$). More generally, although we have called $(0, E_1)$ the low energy region and (E_2, ∞) the high energy region, it may happen that $E_1 > E_2$, in which case we again have a complete description of scattering into direction e_1. We remark that in this case $(\tilde{\Omega}^* - U\Omega^*)E_H((E_2, E_1)) = 0$ for an "explicit" operator U. In fact for an appropriate set of vectors, $Uf = \lim_{t\to\infty} \tilde{U}_0(t)^{-1}U_0(t)f$ may be computed applying Mehler's formula and a stationary phase argument.

In general, there is an energy range, $E_1 < E < E_2$, where we have not constructed wave operators.

Our results overlap those of Hassell, Melrose, and Vasy [HMV1, HMV2, Ha] who consider homogeneous potential scattering in dimension $n = 2$ from a somewhat different point of view. (See [GS] and [Fi].) Additional related work can be found in [Il] and [Yaj].

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2 Classical motion and the Hamilton-Jacobi equation

In this section we assume V is homogeneous of degree zero, $V|S^{n-1}$ is smooth, and V has a non-degenerate local minimum at $e_1 = (1, 0, \dots, 0)$. We normalize V so that $V(e_1) = 0$. Consider a classical orbit, $(x(t), p(t))$, solving Newton's equations

$$\begin{aligned} \frac{dx(t)}{dt} &= p(t), \\ \frac{dp(t)}{dt} &= -\nabla V(x(t)), \end{aligned} \tag{2.1}$$

for which as $t \to \infty$, $x_1(t) \to \infty$, $t^{-1}x_\perp(t) \to 0$, $p_\perp(t) \to 0$. It follows that the energy of the orbit, E, is non-negative. We set $k = \sqrt{2E}$ and assume $k > 0$. Rather than projecting down onto S^{n-1} using the variable $\frac{x(t)}{|x(t)|}$ as in [He], we find it more convenient to introduce the variable $u = \frac{x_\perp}{x_1}$. Introducing a new "time" $\tau = \ln x_1$, we obtain

$$\begin{aligned} \frac{du}{d\tau} &= -u + p_1^{-1} p_\perp, \\ \frac{dp_\perp}{d\tau} &= -p_1^{-1} \nabla_\perp V(1,u), \end{aligned} \qquad p_1 = \sqrt{k^2 - p_\perp^2 - 2V(1,u)}. \tag{2.2}$$

The fact that (2.2) is autonomous is a special property of the system which derives from the homogeneity of V. For various global properties of the system (2.1), see [He]. Here we are interested in motion near the fixed point $u = p_\perp = 0$. We assume that the Hessian of $V(1,u)$ at $u=0$ has only positive eigenvalues $\lambda_2, \ldots, \lambda_n$ and we choose a coordinate system so that the Hessian is given by a diagonal matrix, λ. Then the linearized system is

$$\begin{aligned} \frac{du^{(0)}}{d\tau} &= -u^{(0)} + k^{-1} p_\perp^{(0)}, \\ \frac{dp_\perp^{(0)}}{d\tau} &= -k^{-1} \lambda u^{(0)}. \end{aligned} \tag{2.3}$$

Let $\beta(k)$ and $\tilde{\beta}(k)$ be the diagonal matrices given by

$$\beta(k) = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\lambda/k^2}, \quad \tilde{\beta}(k) = -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\lambda/k^2}. \tag{2.4}$$

If $\beta_j(k)$ is complex, we choose for definiteness

$$\beta_j(k) = -\frac{1}{2} + \frac{i}{2}\sqrt{4\lambda_j/k^2 - 1},$$

and $\tilde{\beta}_j(k)$ its complex conjugate. Then the solutions of (2.3) can be parametrized by vectors w and $\tilde{w}$:

$$\begin{aligned} p_\perp^{(0)} + k\beta(k) u^{(0)} &= e^{\beta(k)\tau} w; \\ p_\perp^{(0)} + k\tilde{\beta}(k) u^{(0)} &= e^{\tilde{\beta}(k)\tau} \tilde{w}. \end{aligned} \tag{2.5}$$

Notice that if we set $E_0 = 2\lambda_{\min}$, $E_2 = 2\lambda_{\max}$, where $\lambda_{\substack{\min \\ \max}} = \substack{\min \\ \max} \{\lambda_2, \ldots, \lambda_n\}$, then if $\frac{k^2}{2} < E_0$ all the eigenvalues β and $\tilde{\beta}$ have a nonzero imaginary part and the orbits of the linearized equations spiral into the center while if $\frac{k^2}{2} > E_2$, β and $\tilde{\beta}$ are negative and there are no spirals. In this case, it will be important to realize that $\beta_j(k) > \tilde{\beta}_\ell(k)$ for all j and ℓ. We define the energy E_1 discussed in the Introduction by setting $\beta_{\max}(k) \equiv -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\lambda_{\min}/k^2} = -\frac{1}{3}$. This gives $E_1 = \frac{9\lambda_{\min}}{4}$.

We introduce a symplectic form on the $2(n-1)$-dimensional phase space of the system (2.2), given by

$$du \wedge dp_\perp = \sum_{j=2}^{n} du_j \wedge dp_j,$$

and note that if $\phi_{\tau,k}$ is the flow for the system (2.2),

$$\phi^*_{\tau,k}(du \wedge dp_\perp) = e^{-\tau} du \wedge dp_\perp. \tag{2.6}$$

Thus, clearly $\phi_{\tau,k}$ is not symplectic but it does preserve Lagrangian manifolds, and this will be important for us when we construct a solution of the Hamilton-Jacobi equation. Eqn. (2.6) can be verified by differentiating the left side and solving the resulting simple differential equation.

Let $\overline{\beta}(k)$ be the diagonal matrix

$$\begin{pmatrix} \beta(k) & 0 \\ 0 & \tilde{\beta}(k) \end{pmatrix}.$$

According to the Sternberg linearization theorem [S1], [S2], [N], the flow $\phi_{\tau,k}$ is conjugate to the linear flow $\phi^{(0)}_{\tau,k}$ of (2.3) via a (local) diffeomorphism ψ_k if k is non-resonant in the sense that there is no relation of the form

$$\overline{\beta}_\ell(k) = \sum_j \alpha_j \overline{\beta}_j(k)$$

for any ℓ and multi-index of non-negative integers α with $|\alpha| \geq 2$. We denote the set of all $k \in (0,\infty)$ for which there is such a relation by $\mathcal{R}$ and remark that some simple considerations show that $\mathcal{R}$ is a discrete set in $(0,\infty)$.

Let $k_2 = \sqrt{4\lambda_{\max}}$. If $k > k_2$, $\overline{\beta}(k)$ is real and we will only be interested in this region in the following. We choose to work in a coordinate system where the linear system is diagonal. Let

$$\begin{aligned} x^{(0)} &= p_\perp^{(0)} + k\beta(k)u^{(0)}, \\ y^{(0)} &= p_\perp^{(0)} + k\tilde{\beta}(k)u^{(0)}; \end{aligned}$$

$$z^{(0)} = \begin{pmatrix} x^{(0)} \\ y^{(0)} \end{pmatrix} = \Lambda(k) \begin{pmatrix} u^{(0)} \\ p_\perp^{(0)} \end{pmatrix};$$

$$\Lambda(k) = \begin{pmatrix} k\beta(k) & I \\ k\tilde{\beta}(k) & I \end{pmatrix}.$$

Then (2.3) becomes

$$\frac{dz^{(0)}}{d\tau} = \overline{\beta}(k) z^{(0)}, \tag{2.7}$$

and with the same change of variable (2.2) becomes

$$\frac{dz}{d\tau} = \overline{\beta}(k)z + G(k,z), \tag{2.8}$$

where for k in any interval with compact closure $\subset (k_2,\infty)$, $G(k,z)$ is C^∞ in all variables for z in a sufficiently small ball, with $|G(k,z)| \le c|z|^2$. Let $\Phi^{(0)}_{\tau,k}$ be the flow associated with (2.7) and $\Phi_{\tau,k}$ the (local) flow associated with (2.8). The Sternberg linearization theorem (generalized slightly to include the parameter k) provides us for each non-resonant $k_0 \in (k_2,\infty)$ an open interval $I \ni k_0$, $I \subset (k_2,\infty)\backslash\mathcal{R}$, an open ball B centered at 0 in $\mathbb{R}^{2(n-1)}$, and a one-parameter family of diffeomorphisms

$$\Psi_k : B \overset{\text{onto}}{\longrightarrow} \Psi_k(B), \quad k \in I,$$

so that on B for $\tau \ge 0$,

$$\Phi_{\tau,k} \circ \Psi_k = \Psi_k \circ \Phi^{(0)}_{\tau,k} = \Psi_k\left(e^{\tau\overline{\beta}(k)}\,\cdot\right).$$

$\Psi_k(z)$ is smooth in all variables including k and satisfies $\Psi_k(0) = 0$, $\Psi_k'(0) = I$. In fact, in the Appendix we show how to construct a single C^∞ function $\Psi(k,z) = \Psi_k(z)$ defined on an open subset of $\mathbb{R} \times \mathbb{R}^{2(n-1)}$ of the form

$$\bigcup_{m=1}^{\infty} \mathcal{O}_m \times B_m,$$

where $\{B_m\}_{m=1}^{\infty}$ is a sequence of open balls centered at 0 in $\mathbb{R}^{2(n-1)}$ and $\mathcal{O}_m$ is a sequence of open subsets of $(k_2,\infty)\backslash\mathcal{R}$ with $\overline{\mathcal{O}}_m \subset \mathcal{O}_{m+1}$, $\overline{\mathcal{O}}_m$ compact, $\mathcal{O}_m \uparrow (k_2,\infty)\backslash\mathcal{R}$. $\Psi_k|B_m$ is a diffeomorphism for each $k \in \mathcal{O}_m$ satisfying the intertwining relation above. See the Appendix for details.

In the new coordinates z, the symplectic form can be written $du \wedge dp_\perp = dz \wedge J\,dz$ where

$$J = \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}; \; A = (2k)^{-1}(\beta(k) - \tilde{\beta}(k))^{-1}.$$

Theorem 2.1 *For $k \in \mathcal{O}_m$, let $\mathcal{M}_k = \{\Psi_k(x^{(0)},0) : (x^{(0)},0) \in B_m\}$. $\mathcal{M}_k$ is a Lagrangian submanifold of the $2(n-1)$-dimensional phase space.*

Remark. Notice we have chosen the Lagrangian subspace $p^{(0)}_\perp + k\tilde{\beta}(k)u^{(0)} = 0$ so that our Lagrangian manifold is made up of orbits of $\phi_{\tau,k}$ which converge as slowly as possible to the fixed point (see (2.5) and note again that $\beta_j(k) \ge \tilde{\beta}_\ell(k)$ for all j and ℓ).

Proof. Consider the Taylor expansion of $\Psi_k(x,0)$,

$$\Psi_k(x,0) \sim \sum_\alpha c_\alpha(k)x^\alpha, \quad c_\alpha(k) \in \mathbb{R}^{2(n-1)}.$$

The coefficients $c_\alpha(k)$ are rational functions of the $\beta_j(k)$ and k ($\tilde{\beta}_j(k) = -1 - \beta_j(k)$) determined by iteration (see [N], [A] or Appendix). Suppose we know that for some $N \geq 1$,

$$\sum_{|\alpha|+|\gamma| \leq N+1} d(c_\alpha x^\alpha) \wedge Jd(c_\gamma x^\gamma) = \sum_{|\alpha|+|\gamma| \leq N+1} \langle c_\alpha, Jc_\gamma \rangle d(x^\alpha) \wedge d(x^\gamma) = 0, \tag{2.9}$$

for a particular k. Let $z_0 = \Psi_k(x, 0)$. Then $F(x) := \Phi_{\tau,k}(z_0) = \Psi_k(e^{\tau\beta(k)}x, 0)$ and

$$\begin{aligned} F^*(dz \wedge Jdz) &= d\Psi_k(e^{\tau\beta}x, 0) \wedge Jd\Psi_k(e^{\tau\beta}x, 0) \\ &= \sum_{|\alpha|+|\gamma| \leq N+1} \langle c_\alpha, Jc_\gamma \rangle e^{\tau\langle \alpha+\gamma, \beta(k)\rangle} d(x^\alpha) \wedge d(x^\gamma) \\ &\quad + \sum_{i,j} a_{ij}(e^{\tau\beta(k)}x) e^{\tau(\beta_i(k)+\beta_j(k))} dx_i \wedge dx_j. \end{aligned} \tag{2.10}$$

It is easy to see that (2.9) implies that for fixed index μ with $|\mu| \leq N+1$,

$$\sum_{\alpha+\gamma=\mu} \langle c_\alpha, Jc_\gamma \rangle d(x^\alpha) \wedge d(x^\gamma) = 0,$$

and thus the first term in (2.10) vanishes. Then using (2.6), (2.10) and the chain rule, we have

$$\begin{aligned} dz_0 \wedge Jdz_0 &= e^\tau F^*(dz \wedge Jdz) \\ &= e^\tau \sum_{i,j} a_{ij}(e^{\tau\beta(k)}x) e^{\tau(\beta_i(k)+\beta_j(k))} dx_i \wedge dx_j. \end{aligned}$$

The coefficient (for fixed x) of $dx_i \wedge dx_j$ is

$$\mathcal{O}\left(e^{\tau((N+2)\beta_{\max}(k)+1)}\right),$$

so if $1 + (N+2)\beta_{\max}(k) < 0$, we obtain

$$dz_0 \wedge Jdz_0 = 0. \tag{2.11}$$

If $N = 1$, (2.9) is trivial, and so we obtain (2.11) if $\beta_{\max}(k) < -\frac{1}{3}$. Expanding $z_0 = \Psi_k(x, 0)$ in a Taylor series and using (2.11), we see that (2.9) holds for any $N \geq 1$ as long as $\beta_{\max}(k) < -\frac{1}{3}$. But since the $c_\alpha(k)$ are rational functions of the components of $\beta(k)$ and k, (2.9) must hold for all non-resonant k and any N. Repeating the above argument we obtain (2.11) for all $k \in \mathcal{O}_m$. □

If we let

$$\psi_k = \Lambda(k)^{-1} \circ \Psi_k \circ \Lambda(k),$$

for $k \in \mathcal{O}_m$, ψ_k is a diffeomorphism of $\Lambda(k)^{-1}B_m$ onto its image, satisfying

$$\phi_{\tau,k} \circ \psi_k = \psi_k \circ \phi^{(0)}_{\tau,k},$$

and $\Lambda(k)^{-1}\mathcal{M}_k = M_k$ is Lagrangian with respect to the symplectic form $du \wedge dp_\perp$.

Proposition 2.2 *If the radii of the B_m are chosen small enough, the manifolds M_k can be parametrized by an equation of the form*

$$p_\perp = \nabla_u f(k,u),$$

where f is defined and smooth on an open set U of $\mathbb{R}^n$ containing $((k_2,\infty)\setminus\mathcal{R})\times\{0\}$. We set $f(k,0)=k$. With this normalization

$$f(k,u) = k - \frac{k}{2}\langle u, \tilde{\beta}(k)u\rangle + \mathcal{O}(|u|^3), \tag{2.12}$$

and the function

$$\overline{S}(k,x) = x_1 f\left(k, \frac{x_\perp}{x_1}\right),$$

defined on $\{(k,x) : x_1 > 0, (k, x_\perp/x_1) \in U\}$, satisfies the Eikonal equation

$$\frac{1}{2}|\nabla_x \overline{S}(k,x)|^2 + V(x) = \frac{k^2}{2}.$$

Proof. Tracing through the change of variables we find that

$$(u,p_\perp) \in M_k \Longleftrightarrow \begin{pmatrix} u \\ p_\perp \end{pmatrix} = \psi_k(u^{(0)}, -k\tilde{\beta}(k)u^{(0)}),$$

with $|\sqrt{k^2-4\lambda}\, u^{(0)}| < r_m$, where r_m is the radius of B_m. Writing

$$\begin{aligned} u &= \pi_1\psi_k(u^{(0)}, -k\tilde{\beta}(k)u^{(0)}) \equiv g_k(u^{(0)}), \\ p_\perp &= \pi_2\psi_k(u^{(0)}, -k\tilde{\beta}(k)u^{(0)}), \end{aligned} \tag{2.13}$$

with π_1 and π_2 the obvious projections, we see that $g_k'(0) = I$ so that if r_m is small enough g_k is a diffeomorphism for all $k \in \mathcal{O}_m$. Thus putting $u^{(0)} = g_k^{-1}(u)$ in the second equation of (2.13) and using the fact that M_k is Lagrangian and the domain of g_k^{-1} is a diffeomorphic image of a ball, we obtain the characterization of M_k as the graph of $\nabla_u f(k,u)$, i.e., $p_\perp = \nabla_k f(k,u)$. From the fact that $\psi_k(u^{(0)}, p_\perp^{(0)}) = \begin{pmatrix} u^{(0)} \\ p_\perp^{(0)} \end{pmatrix}$ + higher order terms, we obtain $\nabla_u f(k,u) = -k\tilde{\beta}(k)u$ plus higher order terms, and thus with the normalization $f(k,0)=k$, (2.12) follows.

To show that the Eikonal equation is satisfied, note that M_k is invariant under $\phi_{\tau,k}$ $(\tau \geq 0)$, so differentiating $p_\perp(\tau) = (\nabla_u f)(k, u(\tau))$ using (2.2) gives

$$\frac{-\nabla_u V(1,u)}{p_1} = f^{(2)}(k,u)(-u + p_1^{-1}p_\perp), \tag{2.14}$$

with $p_\perp = \nabla_u f(k,u)$, and

$$p_1 = \sqrt{k^2 - (\nabla_u f(k,u))^2 - 2V(1,u)}. \tag{2.15}$$

Differentiating p_1^2 using (2.15) and (2.14) results in

$$\nabla_u (p_1 + u \cdot \nabla_u f(k,u) - f(k,u)) = 0.$$

With the normalization $f(k,0) = k$, it follows that

$$p_1 = f(k,u) - u \cdot \nabla_u f(k,u). \tag{2.16}$$

Using the definition of $\overline{S}$ we find with $u = x_\perp / x_1$,

$$\begin{aligned} \partial_{x_1} \overline{S}(k,x) &= f(k,u) - u \cdot \nabla_u f(k,u), \\ \nabla_{x_\perp} \overline{S}(k,x) &= \nabla_u f(k,u), \end{aligned} \tag{2.17}$$

and the Eikonal equation follows from (2.15), (2.16), and (2.17). □

Proposition 2.3 *Suppose U is as in Proposition 2.2 and $(k, u_0) \in U$. Set $p_\perp(0) = \nabla_u f(k, u_0)$, $u(0) = u_0$ so that $(u(0), p_\perp(0)) \in M_k$. Then $(u(\tau), p_\perp(\tau)) \equiv \phi_{\tau,k}(u(0), p_\perp(0))$ satisfies (for $\tau \geq 0$)*

$$p_\perp(\tau) = \nabla_u f(k, u(\tau)),$$

and if $(u^{(0)}(\tau), p_\perp^{(0)}(\tau)) \equiv \psi_k^{-1}(u(\tau), p_\perp(\tau))$, we have

$$\begin{aligned} p_\perp^{(0)}(\tau) + k\beta(k) u^{(0)}(\tau) &= e^{\tau\beta(k)} w, \\ p_\perp^{(0)}(\tau) + k\tilde{\beta}(k) u^{(0)}(\tau) &= 0. \end{aligned}$$

If we set $x_1 = e^\tau$ and define $t = t(\tau)$ up to an additive constant by $\frac{dt}{d\tau} = e^\tau / p_1$, where $p_1 = \sqrt{k^2 - p_\perp(\tau)^2 - 2V(1, u(\tau))}$, then with $x_\perp(t) = x_1(t) u(\tau)$, we have

$$\frac{dx(t)}{dt} = \nabla \overline{S}(k, x(t)).$$

Conversely, suppose $x_1(0) > 0$, $\left(k, \frac{x_\perp(0)}{x_1(0)}\right) \in U$. Then, given the initial condition $x(0) = (x_1(0), x_\perp(0))$ the equation $\frac{dx(t)}{dt} = \nabla_x \overline{S}(k, x(t))$ can be solved for all $t \geq 0$. We have $x_1(t) > 0$ for all $t \geq 0$. Set $\tau = \log x_1(t)$, $u(\tau) = x_\perp(t)/x_1(t)$, $p_\perp(\tau) = \nabla_{x_\perp} \overline{S}(k, x(t))$. Then $(u(\tau), p_\perp(\tau)) \in M_k$ for all $\tau \geq \tau_0 = \log x_1(0)$ and $\phi_{\tau-\tau_0,k}(u(\tau_0), p_\perp(\tau_0)) = (u(\tau), p_\perp(\tau))$.

Proof. The proof is straightforward and is omitted.

We now construct a solution of the time dependent Hamilton-Jacobi equation

$$-\partial_t S(t,x) = \frac{1}{2} |\nabla_x S(t,x)|^2 + V(x), \tag{2.18}$$

from $\overline{S}(k,x)$ using a Legendre transformation. First decrease the radii of the B_m if necessary to make sure that the map

$$\omega_t(k,u) = \frac{kt}{\frac{\partial f}{\partial k}(k,u)}\,(1,u), \quad (k,u) \in U, \tag{2.19}$$

is a diffeomorphism. We define

$$\tilde{U} = \omega_1(U).$$

For (k,x) in the domain of $\overline{S}(k,x)$ $\left(= \left\{(k,x) : x_1 > 0, \left(k, \frac{x_\perp}{x_1}\right) \in U\right\}\right)$, define

$$S(t,x) = \overline{S}(k,x) - \frac{k^2}{2}\,t, \tag{2.20}$$

where

$$t = \frac{1}{k}\,\frac{\partial \overline{S}}{\partial k}\,(k,x),$$

or what is the same thing

$$x_1 = \frac{kt}{\frac{\partial f}{\partial k}(k,u)}, \quad u = \frac{x_\perp}{x_1}.$$

Note that $(k,u) = \omega_t^{-1}(x) = \omega_1^{-1}(x/t)$ gives $k = k(x/t)$ as a smooth function on $\tilde{U}$ and that the domain of S is $\{(t,x) : t > 0, \frac{x}{t} \in \tilde{U}\}$. S is a smooth function on its domain satisfying

$$S(t,x) = tS\left(1, \frac{x}{t}\right), \tag{2.21}$$

and

$$\nabla_x S(t,x) = \nabla_x \overline{S}\left(k\left(\frac{x}{t}\right), x\right),$$

in addition to (2.18).

Proposition 2.4 *Suppose that for $s \geq s_0$, $\overline{x}(s)$ is a solution to $\frac{d\overline{x}(s)}{ds} = \nabla_x \overline{S}(k, \overline{x}(s))$ with $\overline{x}_1(s_0) > 0$ and $\left(k, \frac{\overline{x}_\perp(s_0)}{\overline{x}_1(s_0)}\right) \in U$. Then*

$$t(s) = \frac{1}{k}\,\frac{\partial \overline{S}}{\partial k}\,(k, \overline{x}(s)) = s + c, \tag{2.22}$$

for some constant c. If we define $x(s) = \overline{x}(s-c)$, then

$$\frac{dx(s)}{ds} = \nabla_x S(s, x(s)). \tag{2.23}$$

Conversely, suppose $t_0 > 0$, $\frac{x(t_0)}{t_0} \in \tilde{U}$, and $x(t)$ is a solution to

$$\frac{dx(t)}{dt} = \nabla_x S(t, x(t)), \tag{2.24}$$

for t in some open interval around t_0. Then $x(t)$ extends to a solution of (2.24) for all $t \in [t_0, \infty)$, and if we define k with $k > 0$ and

$$\frac{k^2}{2} = \frac{1}{2}|\nabla_x S(t_0, x(t_0))|^2 + V(x(t_0)),$$

we have $(k, x_\perp(t_0)/x_1(t_0)) \in U$ and $\frac{dx(t)}{dt} = \nabla_x \overline{S}(k, x(t))$.

Proof. Differentiating

$$\frac{1}{2}|\nabla_x \overline{S}(k, x(s))|^2 + V(x(s)) = \frac{k^2}{2},$$

with respect to k gives

$$\nabla_x \frac{\partial \overline{S}}{\partial k}(k, x(s)) \cdot \frac{dx(s)}{ds} = k,$$

while differentiating

$$kt(s) = \frac{\partial \overline{S}}{\partial k}(k, x(s)),$$

with respect to s gives

$$\nabla_x \frac{\partial \overline{S}}{\partial k}(k, x(s)) \cdot \frac{dx(s)}{ds} = k\,\frac{dt(s)}{ds},$$

and thus (2.22) follows. (2.23) then follows from the definition of Legendre transformation which gives $\nabla_x S(t, x) = \nabla_x \overline{S}\left(k\left(\frac{x}{t}\right), x\right)$.

From the definition of $\tilde{U}$ we have $k = k\left(\frac{x(t_0)}{t_0}\right) > 0$, and from Newton's equation which follows from (2.24) we have

$$\frac{k^2}{2} = |\nabla_x S(t, x(t))|^2/2 + V(x(t)),$$

so k is constant. Again, from the definition of Legendre transformation

$$\nabla_x S(t, x(t)) = \nabla_x \overline{S}(k, x(t)).$$

It follows that $\frac{dx(t)}{dt} = \nabla_x \overline{S}(k, x(t))$ and the initial conditions guarantee that $x(t)$ can be extended to $[t_0, \infty)$ as a solution to the latter equation. $\square$

In quantum mechanics, the wave function in $L^2(\mathbb{R}^n)$ must be a function of n commuting observables. To describe the asymptotic motion in the energy regime $k > k_2 = \sqrt{4\lambda_{\max}}$, we choose these observables to be the energy or more specifically k, and a quantum version of the vector $w \in \mathbb{R}^{n-1}$ which occurs in (2.5). As noted previously, the vector $\tilde{w}$ is of less importance in that it describes higher order asymptotics: $\tilde{\beta}_j(k) < \beta_\ell(k)$ for all j and ℓ. In our construction of the Lagrangian manifold necessary to build our solution to the Hamilton-Jacobi equation, we set $\tilde{w} = 0$. Although not necessary, this was convenient.

Given $(t, x) \in \mathbb{R}^{n+1}$ with $t > 0$ and $x/t \in \tilde{U}$, define $w(t, x)$ by solving (2.5), using $\tilde{w} = 0$. We obtain

$$\begin{aligned} w(t,x) &= e^{-\tau\beta(k)}k(\beta(k) - \tilde{\beta}(k))u^{(0)} \\ &= x_1^{-\beta(k)}k(I + 2\beta(k))\pi_1\psi_k^{-1}(u, \nabla_u f(k,u)), \end{aligned}$$

where $k = k(x/t)$, $u = x_\perp/x_1$. $w(t, x)$ is the vector w describing the asymptotics of an orbit $x(s)$ solving $\frac{dx(s)}{ds} = \nabla_x S(s, x(s))$ which goes through x at time t. We can write

$$w(t,x) = t^{-\beta(k(\frac{x}{t}))}g\left(\frac{x}{t}\right).$$

We need to be able to invert the map

$$(t,x) \mapsto (k,w)$$

for fixed t. We have

$$y = g\left(\frac{x}{t}\right) = \left(\frac{k}{\frac{\partial f(k,u)}{\partial k}}\right)^{-\beta(k)} k(I + 2\beta(k))\pi_1\psi_k^{-1}(u, \nabla_u f(k,u)).$$

For small u, we have

$$y = k^{I-\beta(k)}(I + 2\beta(k))u + \mathcal{O}(u^2).$$

Hence, by decreasing the radii r_m if necessary, we can assume this map is invertible and we obtain

$$u = h(k,y)$$

with h smooth. If we write

$$\phi_t(x) = \left(k\left(\frac{x}{t}\right), w(t,x)\right),$$

then ϕ_t is a diffeomorphism mapping $t\tilde{U}$ onto $\tilde{W}_t = \phi_t(t\tilde{U})$. We have

$$\phi_t^{-1}(k,w) = x,$$

with

$$\frac{x}{t} = \left(\frac{k}{\frac{\partial f}{\partial k}(k, h(k, t^{\beta(k)}w))}, \frac{kh(k, t^{\beta(k)}w)}{\frac{\partial f}{\partial k}(k, h(k, t^{\beta(k)}w))}\right).$$

We define an isometric operator $\tilde{U}_0(t)$ on $L^2(\tilde{W}_t)$ as

$$(\tilde{U}_0(t)f)(x) = e^{iS(t,x)}(J(t,x))^{\frac{1}{2}} f\left(k\left(\frac{x}{t}\right), w(t,x)\right), \tag{2.25}$$

where $J(t, x)$ is the Jacobian $\det\phi_t'(x)$. Note that

$$\tilde{U}_0(t) : L^2(\tilde{W}_t) \xrightarrow{\text{onto}} L^2(t\tilde{U})$$

is unitary. The following lemma will be useful when we construct a wave operator using $\tilde{U}_0(t)$:

Lemma 2.5 *If K_1 is a compact subset of $(k_2,\infty)\setminus\mathcal{R}$, let $k_{\max}=\max K_1$. There are open balls B_1 and B_2 centered at $0\in\mathbb{R}^{n-1}$ such that*

$$\tilde{U}\supset K_1\times B_1, \tag{2.26}$$

and with $\epsilon_0=\frac{1}{2}\left(1-\sqrt{1-4\lambda_{\min}/k_{\max}^2}\right)$,

$$\tilde{W}_t\supset K_1\times t^{\epsilon_0}B_2. \tag{2.27}$$

Proof. We have $\omega_1(U)=\tilde{U}$ where ω_1 is a diffeomorphism with $\omega_1(k,0)=(k,0)$. (2.26) follows by a simple compactness argument.

We have $\tilde{W}_t=\phi_t(t\tilde{U})=\phi_t(\omega_t(U))$ and $\phi_t\circ\omega_t(k,u)=(k,w)$ where

$$w=t^{-\beta(k)}\left(\frac{k}{\frac{\partial f}{\partial k}(k,u)}\right)^{-\beta(k)}k(I+2\beta(k))\pi_1\psi_k^{-1}(u,\nabla_u f(k,u)).$$

By a simple compactness argument

$$\phi_1\circ\omega_1(U)\supset K_1\times B_2,$$

for some open ball B_2 centered at 0. By the explicit form of $\phi_t\circ\omega_t$, we have

$$\phi_t\circ\omega_t(U)\supset\{(k,w):\ k\in K_1,\ w\in t^{-\beta(k)}B_2\}.$$

The result then follows by the definition of ϵ_0. □

In some sense, although $\tilde{U}_0(t)$ is not globally defined, it has a generator: If $f\in C_0^\infty((k_2,\infty)\setminus\mathcal{R}\times\mathbb{R}^{n-1}$, we calculate for large enough t (see 2.27),

$$i\partial_t\tilde{U}_0(t)f(x)=\left(H-\frac{1}{2}(p-\nabla_x S(t,x))^2\right)\tilde{U}_0(t)f(x), \tag{2.28}$$

where, of course, $H=\frac{1}{2}p^2+V(x)$. Note that the generator is first order which derives from the fact that $\phi_t(x)$ is constant along the orbit $x(t)$ satisfying $\frac{dx}{dt}=\nabla S(t,x)$.

Generally, we will think of $\tilde{U}_0(t)f$ as belonging to $L^2(\mathbb{R}^n)$ by defining it to be zero outside $t\tilde{U}$.

3 The main theorems

In this section we state our principal results, discuss the Mourre estimate and other estimates from previous work and perform a simple reduction.

We will formulate our theorems for a potential somewhat more general than discussed in the Introduction. Let V_0 and V_1 be real functions on $\mathbb{R}^n$ with $V_0\in C^\infty(\mathbb{R}^n)$ satisfying $x\cdot\nabla V_0(x)=0$ for $|x|>\frac{1}{2}$ and V_1 Laplacian bounded with bound <1 satisfying, in addition, for some $\delta>0$ and as $|x|\to\infty$,

(a) $V_1(x) = \mathcal{O}(|x|^{-1-\delta})$;

(b) $\partial_x^\alpha V_1(x) = \mathcal{O}(|x|^{-2}), \quad |\alpha| = 2$.

We think of V_1 as a short-range perturbation of V_0. Let $V = V_0 + V_1$, and let H be the self-adjoint operator in $L^2(\mathbb{R}^n)$, $n \geq 2$, given by

$$H = -\frac{1}{2}\Delta + V.$$

Let

$$C_r = \{\omega \in S^{n-1} : \ \nabla V_0(\omega) = 0\}.$$

Note that by Sard's theorem, $V_0(C_r)$ has Lebesgue measure zero. We will assume the global condition

$$V_0(C_r) \text{ is at most countable.}$$

To avoid a great deal of cumbersome notation, we formulate our results concerning a single critical point of $V_0|S^{n-1}$, which we assume is $e_1 = (1, 0, \ldots, 0)$. We assume that e_1 is a non-degenerate critical point of $V_0|S^{n-1}$, which is in fact a local minimum. As in the introduction we denote the corresponding projection by P_{e_1} (defined in terms of H). We normalize V_0 so that $V_0(e_1) = 0$ and choose coordinates so that the Hessian of the map $u \mapsto V_0(1, u)$ at $u = 0$ is the diagonal matrix

$$\lambda = \begin{pmatrix} \lambda_2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \quad \lambda_j > 0.$$

Our first theorem is for low energy.

Theorem 3.1 *Let $H_0(t) = \frac{1}{2}p^2 + \frac{1}{2}\langle x_\perp, \lambda x_\perp\rangle/(tp_1)^2$ and suppose $U_0(t)$ is the unitary propagator satisfying*

$$i\partial_t U_0(t) = H_0(t)U_0(t), \qquad U_0(1) = I.$$

Let

$$\begin{aligned} \beta_{\max}(k) &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\lambda_{\min}/k^2}, \\ \lambda_{\min} &= \min\{\lambda_2, \ldots, \lambda_n\}, \end{aligned}$$

and define k_1 by the equation $\beta_{\max}(k_1) = -\frac{1}{3}$, so that $k_1 = \sqrt{\frac{9}{2}\lambda_{\min}}$. Let χ be the indicator function of $[0, k_1]$, and

$$\begin{aligned} \mathcal{H}_1 &= \chi(p_1)L^2(\mathbb{R}^n), \\ \mathcal{H}_2 &= P_{e_1}E_H\left(\left(0, \frac{k_1^2}{2}\right)\right)L^2(\mathbb{R}^n). \end{aligned}$$

Then the strong limit

$$\Omega = \lim_{t\to\infty} e^{itH} U_0(t)$$

exists and defines a unitary operator

$$\Omega : \mathcal{H}_1 \overset{\text{onto}}{\longrightarrow} \mathcal{H}_2,$$

satisfying the intertwining relation

$$e^{itH}\Omega = \Omega e^{\frac{itp_1^2}{2}}.$$

For high energy we have

Theorem 3.2 *Let $S(t,x)$ be the solution to the Hamilton-Jacobi equation*

$$-\partial_t S(t,x) = \frac{1}{2}|\nabla_x S(t,x)|^2 + V_0\left(\frac{x}{|x|}\right)$$

constructed in Section 2, and define

$$\tilde{U}_0(t) : L^2(\tilde{W}_t) \overset{\text{onto}}{\longrightarrow} L^2(t\tilde{U}),$$

as in Eqn. (2.24). Let $k_2 = \sqrt{4\lambda_{\max}}$ and

$$\begin{aligned}
\tilde{\mathcal{H}}_1 &= L^2\left((k_2,\infty)\times\mathbb{R}^{n-1}\right),\\
\tilde{\mathcal{H}}_2 &= P_{e_1} E_H\left(\left(\frac{k_2^2}{2},\infty\right)\right) L^2(\mathbb{R}^n).
\end{aligned}$$

For each $f \in C_0^\infty\left((k_2,\infty)\setminus\mathcal{R}\times\mathbb{R}^{n-1}\right)$ (in the variables (k,w)) the limit

$$\tilde{\Omega} f = \lim_{t\to\infty} e^{itH}\tilde{U}_0(t) f$$

exists and extends by continuity to a unitary operator

$$\tilde{\Omega} : \tilde{\mathcal{H}}_1 \overset{\text{onto}}{\longrightarrow} \tilde{\mathcal{H}}_2,$$

satisfying the intertwining relation

$$e^{itH}\tilde{\Omega} = \tilde{\Omega} e^{itk^2/2}.$$

Remark. With a proper interpretation of the limit on the right hand side an immediate consequence of Theorem 3.2 is

$$(k^+, w^+) := \tilde{\Omega}(k,w)\tilde{\Omega}^* = \lim_{t\to+\infty} e^{itH}(k(t,x), w(t,x))e^{-itH}, \tag{3.1}$$

where the sandwiched vector operators are thought of as vectors of multiplication operators. By the intertwining relation, $k^+ = \sqrt{2H}$. We shall elaborate on the right hand side of (3.1) under an additional condition on the Hessian of V at e_1. The *asymptotic velocity*, $P^+ = (p_1^+, \cdots, p_n^+) = (p_1^+, p_\perp^+)$, would be defined as the vector of commuting self-adjoint operators on $\mathcal{H}_2$ by

$$p_1^+ = s - C_\infty - \lim_{t\to+\infty} e^{itH}\frac{x_1}{t}e^{-itH},$$
$$p_j^+ = s - C_\infty - \lim_{t\to+\infty} e^{itH}t^{\tilde{\beta}_j(t^{-1}x_1)}x_j e^{-itH};\ j = 2, \cdots, n,$$

cf. [DG2, Theorem 4.4.1]. The existence of those operators follows from Theorem 3.2 and the constructions of Section 2 under the additional condition $\lambda_{\max} \le \frac{4}{3}\lambda_{\min}$. (More generally p_1^+ always exists and p_j^+ exists for $j \ge 2$ if $\lambda_j \le \frac{4}{3}\lambda_{\min}$, for example.) For the completely analogous classical asymptotic velocity see [He, Proposition 2.7]; it exhibits the leading asymptotics of scattering orbits nearby e_1. The property called asymptotic absolute continuity in [DG2], which here means the absolute continuity of the above P^+, follows readily from the formula

$$(k^+, w^+) = (p_1^+, (p_1^+)^{-\beta(p_1^+)}(I + 2\beta(p_1^+))p_\perp^+).$$

In the case of small energies, Theorem 3.1 provides the existence of p_1^+, and also of any other component p_j^+ restricted to the energy interval where the corresponding $\beta_j(\sqrt{2E})$ is real-valued, cf. the formulas (4.3) and (4.1) stated below.

A fundamental result in scattering theory, needed in the proof of Theorems 3.1 and 3.2, is a form of the Mourre estimate which we proceed to describe.

Let g be a real function with $1 - g \in C_0^\infty(\mathbb{R}^n)$ and $g = 0$ in a large enough neighborhood of the origin so that the bounds in assumptions (a) through (c) above hold in the support of g. Define the vector field

$$\gamma(x) = \nabla\left(\frac{1}{2}g(x)|x|^2(1 - \eta V_0(x))\right),$$

and the self-adjoint operator

$$A = \frac{1}{2}(p\cdot\gamma(x) + \gamma(x)\cdot p).$$

A is the generator of the group

$$f(\,\cdot\,) \mapsto (j(t,\,\cdot\,))^{\frac{1}{2}}f(\psi_t(\,\cdot\,)),$$

where

$$\begin{aligned}\frac{d}{dt}\psi_t(x) &= \gamma(\psi_t(x)); \qquad \psi_0(x) = x,\\ j(t,x) &= \det\psi_t'(x).\end{aligned}$$

Note that γ is C^∞ and satisfies a global Lipschitz condition so that ψ_t is a global flow: $\psi_t \circ \psi_s = \psi_{t+s}$.

Choosing η small and positive, we have

Lemma 3.3 *[ACH] for any $\lambda \notin V_0(C_r)$ there is an open interval $I \ni \lambda$, a compact operator K, and a positive number c_0 so that*

$$E_H(I)[iH, A]E_H(I) \geq c_0 E_H(I) + K.$$

We omit the proof which involves only slight changes from that of [ACH] to accomodate the possibly singular V_1.

The general theory of [M] and [PSS], and the explicit form of A then give (with $\langle x\rangle = (1 + |x|^2)^{1/2}$)

Lemma 3.4 *The point spectrum of H in $\mathbb{R} \setminus V_0(C_r)$ is a discrete set consisting of eigenvalues of finite multiplicity. H has no singular continuous spectrum. If I is a compact interval disjoint from $V_0(C_r)$ and $\alpha > \frac{1}{2}$, then*

$$\int_{-\infty}^{\infty} \left\|\langle x\rangle^{-\alpha} e^{-itH} E_H(I)\psi\right\|^2 dt \leq C\|\psi\|^2, \tag{3.2}$$

for all $\psi \in$ Ran $P_{\text{cont}}(H)$.

Remarks. (1) We have used the assumption that $V_0(C_r)$ is at most countable to rule out singular continuous spectrum. If V is purely homogeneous of degree zero, this is not necessary [He].

(2) We have made a small improvement to the usual statement of local smoothness of $\langle x\rangle^{-\alpha}$ by allowing the interval I to contain eigenvalues of H. This can be achieved by using the usual Mourre theory for $H + P$ rather than H, where P is the orthogonal projection onto eigenvectors of H with eigenvalue in a neighborhood of a point $\lambda \in I$. The Mourre theory applies because the appropriate bounds on the commutators $[P, A]$ and $[[P, A], A]$ follow from the fact that the eigenvectors in question belong to the domain of multiplication by $\langle x\rangle^2$, which in turn easily follows by the method of [FH].

We will use the following results from [He]:

Lemma 3.5 *If I is a compact interval disjoint from $V_0(C_r)$, then with $\omega = \frac{x}{|x|}$,*

$$\begin{aligned}
\int_{-\infty}^{\infty} \left\|\langle x\rangle^{-\frac{1}{2}} |\nabla V_0(\omega)| e^{-itH} E_H(I)\psi\right\|^2 dt &\leq c\|\psi\|^2, \\
\int_{-\infty}^{\infty} \left\|\langle x\rangle^{-\frac{1}{2}} (p - \omega\langle\omega, p\rangle) e^{-itH} E_H(I)\psi\right\|^2 dt &\leq c\|\psi\|^2,
\end{aligned}$$

for all $\psi \in$ Ran $P_{\text{cont}}(H)$. In addition, for these ψ,

$$\begin{aligned}
\lim_{t\to\infty} |\nabla V_0(\omega)| e^{-itH} E_H(I)\psi &= 0, \\
\lim_{t\to\infty} (p - \omega\langle\omega, p\rangle) e^{-itH} E_H(I)\psi &= 0.
\end{aligned}$$

In proving Lemma 3.5, we use (3.2) and the estimate $V_1(x) = \mathcal{O}(|x|^{-1-\delta})$ to accomodate V_1. Otherwise, the proof is essentially the same as in [He] and need not be repeated.

In our further considerations we would like to treat only V_0 rather than $V_0 + V_1$. That we can do this without loss of generality follows from

Lemma 3.6 *Let $H_0 = -\frac{1}{2}\Delta + V_0$, $H = -\frac{1}{2}\Delta + V$, $V = V_0 + V_1$. Then the wave operator*

$$W = s - \lim_{t\to\infty} e^{itH_0} e^{-itH} P_{\text{cont}}(H)$$

exists and is a unitary operator from Ran $P_{\text{cont}}(H)$ *onto* Ran $P_{\text{cont}}(H_0)$. *In addition,*

$$W : \text{ Ran } P^H_{e_1} \xrightarrow{\text{onto}} \text{Ran } P^{H_0}_{e_1}, \tag{3.3}$$

where $P^H_{e_1}$ is the orthogonal projection onto

$$\left\{\psi : \lim_{t\to\infty} \left\| \left(\frac{x}{|x|} - e_1\right) e^{-itH}\psi \right\| = 0 \right\},$$

and similarly for $P^{H_0}_{e_1}$.

Proof. The proof of the existence and unitarity of W is standard given (3.2) and the fact that $(H_0 + i)^{-1} - (H + i)^{-1}$ is compact [RS]. Eqn. (3.3) follows immediately from the definitions. □

From this point on we set $V_1 = 0$ so that $H = -\frac{1}{2}\Delta + V_0$. In our proof of completeness of the wave operators, we will use another simplifying reduction. We will modify V_0 to produce another potential $\tilde{V}_0 \in C^\infty(\mathbb{R}^n)$ with the following properties:

(i) $\tilde{V}_0$ is homogeneous of degree zero for $|x| > \frac{1}{2}$;

(ii) $\tilde{V}_0|S^{n-1} = V_0|S^{n-1}$ in a neighborhood of e_1;

(iii) $-\partial_1\tilde{V}_0(x) > 0$ on $S^{n-1} \setminus \{e_1, -e_1\}$;

(iv) $\tilde{V}_0\left(\frac{x}{|x|}\right) > L$ if $\frac{x_1}{|x|} \le \frac{1}{\sqrt{2}}$;

(v) $\tilde{V}_0(x) \ge 0$.

Here L is any preassigned number which will be chosen larger than the maximum energy of the range we are working in. We have chosen the cone $\left\{x : \frac{x_1}{|x|} > \frac{1}{\sqrt{2}}\right\}$ arbitrarily.

To produce $\tilde{V}_0$, choose $\delta > 0$ small so that for $1 > \frac{x_1}{r} \ge 1 - \delta$, $-\partial_1 V_0(\omega) > 0$ where $\omega = \frac{x}{r}$, $r = |x|$. Let $\chi \in C^\infty(\mathbb{R})$ with $0 \le \chi \le 1$, $\chi(t) = 1$ if $1 \ge \frac{x_1}{r} > 1 - \frac{\delta}{2}$, $\chi(t) = 0$ if $\frac{x_1}{r} \le 1 - \delta$. Let $\gamma \in C^\infty(\mathbb{R})$ with $\gamma(t) = 0$ if $t \ge 1 - \frac{\delta}{4}$, $\gamma'(t) \le 0$, and $\gamma'(t) \le -1$ if $t \le 1 - \frac{\delta}{2}$. For large μ, let

$$\tilde{V}_0(\omega) = \chi\left(\frac{x_1}{r}\right) V_0(\omega) + \mu\gamma\left(\frac{x_1}{r}\right),$$

and extend $\tilde{V}_0$ to $\mathbb{R}^n$ so that it is smooth, non-negative, and homogeneous of degree zero for $|x| > \frac{1}{2}$. If δ is chosen small enough and μ large enough, it is easy to verify (i)–(iv). Since $V_0(e_1) = 0$, clearly $\tilde{V}_0|S^{n-1} \geq 0$. It is easy to extend $\tilde{V}_0$ to $\mathbb{R}^n$ so that (v) is true.

The reason V_0 can be replaced by $\tilde{V}_0$ is the existence and unitarity of the relevant wave operator:

Lemma 3.7 *Suppose V_0 and $\tilde{V}_0$ are as above. Let $H = -\frac{1}{2}\Delta + V_0$ and $\tilde{H} = -\frac{1}{2}\Delta + \tilde{V}_0$. Then the wave operator*

$$\tilde{W} = s - \lim_{t\to\infty} e^{itH} e^{-it\tilde{H}}$$

exists on Ran $P_{e_1}^{\tilde{H}}$ *and defines a unitary map*

$$\tilde{W} : \text{Ran } P_{e_1}^{\tilde{H}} \xrightarrow{\text{onto}} \text{Ran } P_{e_1}^{H}.$$

Proof. As for Lemma 3.6, the proof is standard once it is realized that for bounded continuous f, $(f(H) - f(\tilde{H}))e^{-it\tilde{H}} \to 0$ strongly on Ran $P_{e_1}^{\tilde{H}}$ and similarly with H and $\tilde{H}$ reversed. □

4 Existence of wave operators

In this section all the assumptions made in Section 3 are in force, but because of Lemma 3.6 we drop V_1 from consideration and write $H = -\frac{1}{2}\Delta + V_0$.

Theorem 4.1 *Let $H_0(t) = \frac{1}{2}p^2 + \frac{\langle x_\perp, \lambda x_\perp\rangle}{2(p_1 t)^2}$, and suppose $U_0(t)$ is the unitary propagator satisfying*

$$i\frac{\partial U_0(t)}{\partial t} = H_0(t)U_0(t), \quad U_0(1) = I.$$

Let

$$\begin{aligned} \beta_{\max}(k) &= -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\lambda_{\min}/k^2} \\ \lambda_{\min} &= \min\{\lambda_2, \dots, \lambda_n\}, \end{aligned}$$

and define $k_1 > 0$ by the equation $\beta_{\max}(k_1) = -\frac{1}{3}$ (i.e., $k_1 = \sqrt{\frac{9}{2}\lambda_{\min}}$). Let χ be the indicator function of $[0, k_1]$ and $\mathcal{H}_1 = \chi(p_1)L^2(\mathbb{R}^n)$. Then the strong limit

$$\Omega = \lim_{t\to\infty} e^{itH} U_0(t)$$

exists on $\mathcal{H}_1$. We have the intertwining relation

$$e^{itH}\Omega = \Omega e^{\frac{itp_1^2}{2}}.$$

Proof. We prove convergence on a dense subset of $\mathcal{H}_1$, namely for those $f \in \mathcal{H}_1$, whose Fourier transform, $\hat{f}$, is in $C_0^\infty((0,k_1)\backslash\mathcal{E}\times\mathbb{R}^{n-1})$ with $\mathcal{E} = \{2\sqrt{\lambda_2}, 2\sqrt{\lambda_3}, \dots, 2\sqrt{\lambda_n}\}$. We define

$$\begin{aligned} x_\perp(t) &= U_0(t)^{-1} x_\perp U_0(t); \\ p_\perp(t) &= U_0(t)^{-1} p_\perp U_0(t); \\ x_1(t) &= U_0(t)^{-1} x_1 U_0(t); \\ u(t) &= x_\perp(t)/t. \end{aligned}$$

Using the simple linear differential equations satisfied by these operators we obtain

$$u(t) = \left(1 - 4\lambda p_1^{-2}\right)^{-\frac{1}{2}} \left(t^{\beta(p_1)} w - t^{\tilde{\beta}(p_1)} \tilde{w}\right) \tag{4.1}$$

$$p_\perp(t) = \left(1 - 4\lambda p_1^{-2}\right)^{-\frac{1}{2}} \left(\beta(p_1) t^{\tilde{\beta}(p_1)} \tilde{w} - \tilde{\beta}(p_1) t^{\beta(p_1)} w\right) \tag{4.2}$$

$$x_1(t) = x_1 + (t-1)p_1 - \int_1^t p_1^{-3} \langle u(s), \lambda u(s)\rangle ds, \tag{4.3}$$

where β and $\tilde{\beta}$ are defined in Section 2 and

$$w = p_\perp + \beta(p_1) x_\perp, \quad \tilde{w} = p_\perp + \tilde{\beta}(p_1) x_\perp.$$

Note that the operators in (4.1)–(4.3) are well defined even when $p_1 \in \mathcal{E}$ (using a limiting procedure).

In order to prove existence of Ω we will need to see where $U_0(t)f$ is localized for large t.

Lemma 4.2 *Suppose $\chi \in C_0^\infty(\mathbb{R})$ and $\chi = 1$ in a neighborhood of 0. Suppose α and γ are multi-indices. Then for any non-negative integers m and N and some $\delta > 0$,*

$$\left\| (x_\perp)^\alpha \left(1 - \chi\left(p_1 - \frac{x_1}{t}\right)\right) U_0(t) f \right\| \leq C_N t^{-N} \tag{4.4}$$

$$\left\| \left(p_1 - \frac{x_1}{t}\right)^m (p_\perp)^\gamma \left(\frac{x_\perp}{t}\right)^\alpha U_0(t) f \right\| \leq C_m t^{-(\frac{1}{3}+\delta)(|\alpha|+|\gamma|)} t^{-\frac{2}{3}m}. \tag{4.5}$$

Proof of Lemma 4.2. The estimate (4.5) follows directly from (4.1)–(4.3). Notice that factors of x_1/t originating from the right side of (4.3) are harmless. They lead to differentiation in the p_1 variable which in turn gives at most harmless powers of $\ln t$. The extra δ in (4.5) arises from the fact that $\hat{f}$ has compact support in $(0,k_1) \times \mathbb{R}^{n-1}$. To prove (4.4) we estimate $|1 - \chi(s)| \leq c'_M |s|^M$, and then use (4.5). □

In the following we use $\mathcal{O}(t^{-\infty})$ to mean $\mathcal{O}(t^{-N})$ for any N.

Lemma 4.3 *Suppose* $\chi_j \in C^\infty(\mathbb{R})$ *with* $\chi_j' \in C_0^\infty(\mathbb{R})$ *and* $\operatorname{supp}\chi_j \subset (0,\infty)$, $j = 1,2,3$. *In addition, suppose* $\epsilon_j \in \mathbb{R}$ *with*

$$\epsilon_2 + \epsilon_3 > \epsilon_1.$$

Then for $t > 0$*:*

$$\left\| \chi_1\left(\epsilon_1 - \frac{x_1}{t}\right)\chi_2\left(\frac{x_1}{t} - p_1 - \epsilon_2\right)\chi_3\left(p_1 - \epsilon_3\right)\right\| = \mathcal{O}(t^{-\infty}).$$

Proof of Lemma 4.3. By [Fo, Corollary 2.19] the Weyl symbol of the operator $\chi_2\left(\frac{x_1}{t} - p_1 - \epsilon_2\right)$ is $\chi_2\left(\frac{x_1}{t} - \xi_1 - \epsilon_2\right)$ and thus all three operators have symbols in the class $S(1,g)$ with $g = \langle t\rangle^{-2}dx_1^2 + d\xi_1^2$. In addition the product of three symbols or of their derivatives vanishes so the bound follows from the calculus [Hö, Theorems 18.5.4 and 18.6.3]. □

We are now ready to complete the proof of Theorem 4.1. We use Cook's method and therefore we need to show

$$\left\|\left(V_0(x) - \frac{\langle x_\perp, \lambda x_\perp\rangle}{(tp_1)^2}\right)U_0(t)f\right\| = F(t)$$

is integrable on $[1,\infty)$.

For some $\epsilon_3 > 0$, $\operatorname{supp}\hat{f}(\xi_1,\cdot) \subset (2\epsilon_3,\infty)$. We choose $\chi_+ \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\chi_+ \subset (1,\infty)$ and $\operatorname{supp}(1-\chi_+) \subset (-\infty,2)$. Let $\chi_- = 1 - \chi_+$. We have $\chi_-\left(\frac{p_1}{\epsilon_3}\right)U_0(t)f = 0$, and

$$\begin{aligned}
&\left\|\left(V_0(x) - \frac{\langle x_\perp, \lambda x_\perp\rangle}{tp_1)^2}\right)U_0(t)f\right\| \\
\leq\ & \left\|\chi_+\left(\frac{4x_1}{\epsilon_3 t}\right)\left(V_0(x) - \frac{\langle x_\perp, \lambda x_\perp\rangle}{(tp_1)^2}\right)\chi_+\left(\frac{p_1}{\epsilon_3}\right)U_0(t)f\right\| + \left\|V_0(x)\chi_-\left(\frac{4x_1}{\epsilon_3 t}\right)U_0(t)f\right\| \\
+\ & \left\|\left[\chi_-\left(\frac{4x_1}{\epsilon_3 t}\right), \frac{1}{p_1^2}\chi_+\left(\frac{p_1}{\epsilon_3}\right)\right]\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}U_0(t)f\right\| \\
+\ & c\left\|\chi_-\left(\frac{4x_1}{\epsilon_3 t}\right)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}U_0(t)f\right\|.
\end{aligned} \tag{4.6}$$

Consider the last term. We have for $\chi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\chi \subset (-1,1)$, and $\chi = 1$ in $\left(-\frac{1}{2},\frac{1}{2}\right)$,

$$\begin{aligned}
&\left\|\chi_-\left(\frac{4x_1}{\epsilon_3 t}\right)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}U_0(t)f\right\| \\
\leq\ & \left\|\chi_-\left(\frac{4x_1}{\epsilon_3 t}\right)\chi\left(\frac{4\left(\frac{x_1}{t} - p_1\right)}{\epsilon_3}\right)\chi_+\left(\frac{p_1}{\epsilon_3}\right)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}U_0(t)f\right\|
\end{aligned}$$

$$+ \quad c\left\| \frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}\left(1-\chi\left(\frac{4\left(\frac{x_1}{t}-p_1\right)}{\epsilon_3}\right)\right)U_0(t)f\right\|. \tag{4.7}$$

According to Lemma 4.3 and Eqn. (4.5) of Lemma 4.2, the first term of (4.7) is $\mathcal{O}(t^{-\infty})$ and according to Lemma 4.2, Eqn. (4.4) the same is true for the second term.

The second term on the right side of (4.6) is also $\mathcal{O}(t^{-\infty})$ by the same argument. The commutator in the third term has operator norm $= \mathcal{O}(t^{-1})$ so that according to Lemma 4.2 this term is $\mathcal{O}\left(t^{-1-\frac{2}{3}}\right)$, hence integrable.

For the first term on the right of (4.6) we write

$$V_0(x) - \frac{\langle x_\perp, \lambda x_\perp\rangle}{2(tp_1)^2} = \left(V_0\left(1, \frac{x_\perp}{x_1}\right) - \frac{\langle x_\perp, \lambda x_\perp\rangle}{2x_1^2}\right) + \left(\frac{\langle x_\perp, \lambda x_\perp\rangle}{2x_1^2} - \frac{\langle x_\perp, \lambda x_\perp\rangle}{2(p_1 t)^2}\right). \tag{4.8}$$

The first term in (4.8) can be bounded by $c\left(\frac{|x_\perp|}{t}\right)^3$ in the support of $\chi_+\left(\frac{4x_1}{\epsilon_3 t}\right)$ and thus using Lemma 4.2, Eqn. (4.5), this gives an integrable term. Note that this is the only place we use the full force of the cut-off at k_1.

For the second term of (4.8) we write

$$\begin{aligned}&\chi_+\left(\tfrac{4x_1}{\epsilon_3 t}\right)\left(\tfrac{\langle x_\perp,\lambda x_\perp\rangle}{x_1^2} - \tfrac{\langle x_\perp,\lambda x_\perp\rangle}{(p_1 t)^2}\right)\chi_+\left(\tfrac{p_1}{\epsilon_3}\right)\\ &\quad = \chi_+\left(\tfrac{4x_1}{\epsilon_3 t}\right)\left(\tfrac{t}{x_1}\right)^2\left(p_1^2 - \left(\tfrac{x_1}{t}\right)^2\right)\tfrac{1}{p_1^2}\chi_+\left(\tfrac{p_1}{\epsilon_3}\right)\tfrac{\langle x_\perp,\lambda x_\perp\rangle}{t^2}.\end{aligned}$$

Using

$$p_1^2 - \left(\frac{x_1}{t}\right)^2 = p_1\left(p_1 - \frac{x_1}{t}\right) + \frac{x_1}{t}\left(p_1 - \frac{x_1}{t}\right) - it^{-1}$$

and Lemma 4.2, Eqn. (4.5), we see that this term contributes $\mathcal{O}\left(t^{-\frac{4}{3}}\right)$. Thus existence of the wave operator is proved.

To prove the intertwining property we write

$$\overline{U}_0(t) = e^{ip_1^2(t-1)/2}U_0(t)$$

and compute

$$\overline{U}_0(t)^{-1}\overline{U}_0(t+s) = I - i\int_0^s \overline{U}_0(t)^{-1}\left(\frac{p_\perp^2}{2} + \frac{\langle x_\perp, \lambda x_\perp\rangle}{2p_1^2(t+\theta)^2}\right)\overline{U}_0(t+\theta)d\theta,$$

which gives

$$\begin{aligned} e^{-isH}\Omega f &= \lim_{t\to\infty}\left(e^{itH}U_0(t)\right)U_0(t)^{-1}U_0(t+s)f \\ &= \Omega e^{-isp_1^2/2}f - i\lim_{t\to\infty}\left(e^{itH}U_0(t)\right)e^{-isp_1^2/2} \\ &\quad \cdot \int_0^s \overline{U}_0(t)^{-1}\left(\frac{p_\perp^2}{2} + \frac{\langle x_\perp, \lambda x_\perp\rangle}{2p_1^2(t+\theta)^2}\right)\overline{U}_0(t+\theta)f\, d\theta.\end{aligned}$$

The last term is zero by Lemma 4.2. □

We now turn to the high energy regime.

Theorem 4.4 *Let $\tilde{\mathcal{H}}_1 = L^2((k_2,\infty)\times\mathbb{R}^{n-1})$ and define $\tilde{U}_0(t)$ as in (2.25). Then the limit*

$$\tilde{\Omega} f = \lim_{t\to\infty} e^{itH}\tilde{U}_0(t) f$$

exists for $f \in C_0^\infty((k_2,\infty)\backslash\mathcal{R}\times\mathbb{R}^{n-1})$ and extends by continuity to an isometric operator on $\tilde{\mathcal{H}}_1$. We have

$$e^{itH}\tilde{\Omega} = \tilde{\Omega} e^{\frac{itk^2}{2}}. \tag{4.9}$$

Proof. Using (2.28) we calculate for large t,

$$i\partial_t e^{itH}\tilde{U}_0(t) f(x) = e^{itH} e^{iS}\frac{1}{2}\Delta_x\left(J^{\frac{1}{2}}(t,x) f\left(k\left(\frac{x}{t}\right), w(t,x)\right)\right),$$

where we have used Lemma 2.5 to conclude that $f \in L^2(\tilde{W}_t)$ for large t. Using Cook's method, it suffices to show

$$\int_{T_0}^\infty \left\|\Delta_x J^{\frac{1}{2}}(t,x) f\left(k\left(\frac{x}{t}\right), w(t,x)\right)\right\| dt < \infty. \tag{4.10}$$

Let $\pi_1(\text{supp } f)$ be the projection of supp f onto the first factor of $\mathbb{R}\times\mathbb{R}^{n-1}$, and define

$$\begin{aligned} k_{\min} &= \min \pi_1(\text{supp } f), \\ k_{\max} &= \max \pi_1(\text{supp } f), \\ \tilde{\beta}_{\max} &= -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\lambda_{\max}/k_{\min}^2}. \end{aligned}$$

We must estimate the quantities

$$\partial_{x_j}\phi_t(x), \quad \frac{\partial_{x_j} J}{J}, \quad \Delta_x\phi_t(x), \quad \frac{\Delta_x J}{J}$$

for $\phi_t(x) \in \text{supp } f$. We have

$$\left|\partial_{x_j} k\left(\frac{x}{t}\right)\right| \le ct^{-1}, \quad \left|\Delta_x k\left(\frac{x}{t}\right)\right| \le ct^{-2}.$$

As in Section 2, we can write

$$w(t,x) = t^{-\beta\left(k\left(\frac{x}{t}\right)\right)} g\left(\frac{x}{t}\right),$$

where

$$g\left(\frac{x}{t}\right) = \left(\frac{k}{\frac{\partial f(k,u)}{\partial k}}\right)^{-\beta(k)} k(I + 2\beta(k))\pi_1\psi_k^{-1}(u, \nabla_u f(k,u)),$$

$$u = \frac{x_\perp}{x_1}, \quad k = k\left(\frac{x}{t}\right).$$

We have

$$\partial_{x_j} w(t,x) = -\frac{(\partial_j k)\left(\frac{x}{t}\right)}{t}\,\beta'(k)\ln t\, w(t,x) + t^{\tilde\beta(k)}(\partial_j g)\left(\frac{x}{t}\right),$$

so that

$$|\partial_{x_j} w(t,x)| \le ct^{\tilde\beta_{\max}},$$

since $w(t,x)$ is bounded in supp f.

Similarly,

$$|\Delta_x w(t,x)| \le c\left(\frac{\ln t}{t}\right) t^{\tilde\beta_{\max}},$$

so that

$$|\nabla_x \phi_t(x)| \le ct^{\tilde\beta_{\max}},\ \ |\Delta_x \phi_t(x)| \le \frac{c\ln t}{t}\cdot t^{\tilde\beta_{\max}}.$$

$J(t,x) = \det \phi_t'(x)$, but

$$(\phi_t \circ \omega_t)(k,u) = (k,w),$$

where w is given by

$$w = t^{-\beta(k)}\left(\frac{k}{\frac{\partial f(k,u)}{\partial k}}\right)^{-\beta(k)} k(I+2\beta(k))\pi_1\psi_k^{-1}(u,\nabla_u(f(k,u)).$$

It follows that

$$\det(\phi_t\circ\omega_t)'(k,u) = t^{-\operatorname{tr}\beta(k)}\alpha(k,u),$$

for some smooth function α, while

$$\det \omega_t'(k,u) = t^n\ \tilde\alpha(k,u),$$

where $\tilde\alpha(k,u)$ is smooth and bounded away from zero for $\phi_t\circ\omega_t(k,u)\in$ supp f. Thus $J(t,x) = t^{-n}t^{-\operatorname{tr}\beta(k)}\alpha(k,u)/\tilde\alpha(k,u)$. Differentiation gives (for $\phi_t(x)\in$ supp f)

$$\begin{aligned}|\nabla_x \ln J(t,x)| &\le \frac{c\ln t}{t};\\ |\Delta_x \ln J(t,x)| &\le \frac{c\ln t}{t^2}.\end{aligned}$$

Putting the estimates together gives

$$\left\|\Delta_x\left(J^{\frac12}(t,x)f\left(k\left(\frac{x}{t}\right),w(t,x)\right)\right)\right\|_{L^2} \le ct^{2\tilde\beta_{\max}},$$

where the largest term comes from

$$\sum_{k,\ell}\left\|J^{\frac12}(t,x)\partial_k\partial_\ell f(\phi_t(x))\nabla\phi_t(x)_\ell\cdot\nabla\phi_t(x)_k\right\|_{L^2}.$$

Since $2\tilde{\beta}_{\max} < -1$, (4.10) is valid proving existence of $\tilde{\Omega}$. By the definition of $\tilde{U}_0(t)$ and the strong convergence, $\|\tilde{\Omega} f\| = \|f\|$, and thus $\tilde{\Omega}$ extends by continuity to an isometry on $\tilde{\mathcal{H}}_1$ (into $L^2(\mathbb{R}^n)$).

To prove (4.9), note

$$e^{isH}\tilde{\Omega} f = \lim_{t\to\infty} e^{itH}\tilde{U}_0(t-s)f,$$

and for $f \in C_0^\infty((k_2,\infty)\setminus\mathcal{R}\times\mathbb{R}^{n-1})$,

$$\tilde{U}_0(t-s)f(x) = e^{iS(x,t-s)}J^{\frac{1}{2}}(t-s,x)f(\phi_{t-s}(x)).$$

We have

$$\begin{aligned} S(x,t-s) &= S(x,t) - s\frac{\partial S}{\partial t}(x,t) + \mathcal{O}(t^{-1}) \\ &= S(x,t) + sk\left(\frac{x}{t}\right)^2/2 + \mathcal{O}(t^{-1}) \end{aligned}$$

from the Hamilton-Jacobi equation. From the functional form of $J^{\frac{1}{2}}(t,x)f(\phi_t(x))$ we have

$$\left(\partial_t + \left(\frac{\nabla_x\cdot\nabla_x S(t,x) + \nabla_x S(t,s)\cdot\nabla_x}{2}\right)\right)J^{\frac{1}{2}}(t,x)f(\phi_t(x)) = 0,$$

and using the previous estimates we obtain for fixed s,

$$\|J^{\frac{1}{2}}(t-s,x)f(\phi_{t-s}(x)) - J^{\frac{1}{2}}(t,x)f(\phi_t(x)\|_{L^2} \le ct^{\tilde{\beta}_{\max}}.$$

Thus

$$\left\|\tilde{U}_0(t-s)f - \tilde{U}_0(t)\left(e^{\frac{isk^2}{2}}f\right)\right\|_{L^2} \le ct^{\tilde{\beta}_{\max}},$$

which gives (4.9). □

5 Localization of $e^{-it\tilde{H}}\psi$

In this section, $\tilde{H} = -\frac{1}{2}\Delta + \tilde{V}_0$ where $\tilde{V}_0$ was defined in Section 3 depending on a large energy L, at our disposal. We will make sure that all our energy localizations are carried out with $f(\tilde{H})$ such that $f \in C_0^\infty((-\infty, L))$. The notation χ_S is used to signify the characteristic function of a set S.

Proposition 5.1 *Suppose $f \in C_0^\infty((-\infty,L))$. Then for any N_1 and N_2, the operator*

$$\langle x\rangle^{N_1}\chi_{\left[-1,\frac{1}{\sqrt{2}}\right]}\left(\frac{x_1}{|x|}\right)f(\tilde{H})\langle x\rangle^{N_2}$$

is bounded.

Proof. Since $\tilde{V}_0(\omega) > L$ if $\frac{x_1}{|x|} \leq \frac{1}{\sqrt{2}}$, we can choose $\chi_1 \in C_0^\infty\left(\left(\frac{1}{\sqrt{2}}, \infty\right)\right)$ and non-negative so that

$$\tilde{V}_0(\omega) + \chi_1\left(\frac{x_1}{|x|}\right) > L$$

for all x. We choose $\chi_3 \in C^\infty(\mathbb{R}^n)$ and $\chi_2 \in C_0^\infty(\mathbb{R}^n)$ with $1 - \chi_3 \in C_0^\infty(\mathbb{R}^n)$, $\chi_3, \chi_2 \geq 0$, and $\chi_3 = 0$ in a neighborhood of 0 so that for all $x \in \mathbb{R}^n$,

$$V_2(x) \equiv \tilde{V}_0(x) + \chi_1\left(\frac{x_1}{|x|}\right)\chi_3(x) + \chi_2(x) > L.$$

Let $H_2 = -\frac{1}{2}\Delta + V_2$. Then $f(H_2) = 0$. If $\tilde{f}$ is an almost analytic extension of f (see [DG2, Appendix C], for example), then

$$f(\tilde{H}) = \pi^{-1}\int \overline{\partial}\tilde{f}(z)((\tilde{H}-z)^{-1} - (H_2 - z)^{-1})d^2z,$$

so that with $k(x) = \chi_1\left(\frac{x_1}{|x|}\right)\chi_3(x)$,

$$\begin{aligned} f(\tilde{H}) &= \pi^{-1}\int \overline{\partial}\tilde{f}(z)(\tilde{H}-z)^{-1}k(x)(H_2-z)^{-1}d^2z \\ &\quad + \pi^{-1}\int \overline{\partial}\tilde{f}(z)\tilde{H}-z)^{-1}\chi_2(x)(H_2-z)^{-1}d^2z. \end{aligned} \tag{5.1}$$

In the first term of (5.1) we repeatedly move $k(x)$ to the left using

$$\begin{aligned} &[(\tilde{H}-z)^{-1}, k(x)] \\ &= \sum_{j=1}^{\ell}(-1)^j ad_{\tilde{H}}^j(k(x))(\tilde{H}-z)^{-(j+1)} + (-1)^{\ell+1}(\tilde{H}-z)^{-1}ad_{\tilde{H}}^{\ell+1}(k(x))(H-z)^{-(\ell+1)}, \end{aligned}$$

and note that

$$\chi_{\left[-1,\frac{1}{\sqrt{2}}\right]}\left(\frac{x_1}{|x|}\right) ad_{\tilde{H}}^j(k(x)) = 0. \tag{5.2}$$

We calculate

$$ad_{\tilde{H}}^{\ell+1}(k(x)) = \sum_{|\alpha|\leq\ell+1} p^\alpha k_\alpha(x),$$

where $k_\alpha \in C^\infty(\mathbb{R}^n)$ with $\partial^\beta k_\alpha(x) = \mathcal{O}(|x|^{-|\beta|-\ell-1})$ and $p = -i\nabla$. A simple induction gives for any integer m,

$$\|(p^2+1)\langle x\rangle^{-m}(\tilde{H}-z)^{-1}\langle x\rangle^m\| \leq c_m|\mathrm{Im}\ z|^{-N} \tag{5.3}$$

for some N. (5.3) and similar estimates, along with (5.2), gives for any integers ℓ_1, ℓ_2, with $\ell_j \geq 0$, $\ell_1 + \ell_2 = \ell$,

$$\left\|\langle x\rangle^{\ell_1}\chi_{\left[-1,\frac{1}{\sqrt{2}}\right]}\left(\frac{x_1}{|x|}\right)\int \overline{\partial}\tilde{f}(z)(\tilde{H}-z)^{-1}k(x)(H_2-z)^{-1}d^2z\langle x\rangle^{\ell_2}\right\| < \infty.$$

A similar but easier argument works for the second term in (5.1). Thus taking $\ell = N_1 + N_2$, $\ell_1 = N_1$, $\ell_2 = N_2$ we obtain the desired result. $\square$

In what follows we take $\phi \in \mathcal{S}(\mathbb{R}^n)$, and write $\psi_t = e^{-it\tilde{H}} f(\tilde{H})\phi$.

Proposition 5.2 *Suppose $f \in C_0^\infty((0,L) \setminus \sigma_{pp}(\tilde{H}))$. Then there is a $\lambda_0 > 0$ so that*

$$\left\| \chi_{[0,\lambda_0]}\left(\frac{|x|}{t}\right)\psi_t \right\| = \mathcal{O}(t^{-\infty}). \tag{5.4}$$

Proof. It is easy to see that if $\delta > 0$ and

$$A_1 = \frac{x_1 p_1 + p_1 x_1}{2} + \frac{\delta}{2}(x \cdot p + p \cdot x),$$

then for any $\lambda \in (0,L) \setminus \sigma_{pp}(\tilde{H})$ we have a Mourre estimate:

$$g(\tilde{H})[i\tilde{H}, A_1]g(\tilde{H}) \geq c_0 g(\tilde{H})^2, \tag{5.5}$$

if $g \in C_0^\infty(\mathbb{R})$ has support in a small enough interval around λ. To see (5.5) we compute

$$[i\tilde{H}, A_1] = p_1^2 + \delta|p|^2 - x_1\partial_1\tilde{V}_0(x) - \delta x \cdot \nabla\tilde{V}_0(x),$$

and note that we can choose $\delta_1 \in (0, 2\delta)$ so that if $x_1 > \frac{1}{2}$,

$$-x_1\partial_1\tilde{V}_0(x) \geq \delta_1\tilde{V}_0(x).$$

Since $\tilde{V}_0(x) > L$ in the region $\{x : x_1 \leq \frac{1}{2}, |x| > 1\}$, the region where $-x_1\partial_1\tilde{V}_0(x) < \delta_1\tilde{V}_0(x)$ contributes a compact term to the left side of (5.5) if the support of g is small enough, cf. Proposition 5.1. The remainder of the argument to obtain (5.5) is standard.

One may now invoke either the proof of (5.4) in [Sk, Examples 1 and 2] or the one in [DG2, p. 193–197]. Only small modifications are needed in the proof, which in both cases originates from [SS]. Note that $2A_1$ is the Heisenberg derivative of $\langle x, x\rangle_\delta = x_1^2(1+\delta) + |x_\perp|^2\delta$. (A different conjugate operator yielding a proof of (5.4) along the same line, although a slightly more complicated one, would be the A of Lemma 3.3.) $\square$

Proposition 5.3 *Suppose $f \in C_0^\infty\left(\left(-\infty, \frac{b^2}{2}\right)\right)$ where $\frac{b^2}{2} \leq L$. Then for any $\Lambda_0 \geq |b|$ and any $N \geq 0$,*

$$\left\| \left(\frac{|x|}{t}\right)^N \chi_{[\Lambda_0,\infty)}\left(\frac{|x|}{t}\right)\psi_t \right\| = \mathcal{O}(t^{-\infty}). \tag{5.6}$$

Proof. We mimic either [Sk, Example 3] or [DG2, p. 190–192]. $\square$

Proposition 5.4 *Suppose $f \in C_0^\infty\left(\left(0, \frac{b^2}{2}\right) \setminus \sigma_{pp}(\tilde{H})\right)$ where $\frac{b^2}{2} \leq L$. Then there is an $R > 0$ so that for all $l \in \mathbb{N} \cup 0$ and all bounded functions g with $g(\xi) = 0$ for $|\xi| < R$,*

$$\|\langle p\rangle^l g(p)\psi_t\| = \mathcal{O}(t^{-\infty}).$$

Proof. Assume first that $l = 0$. According to [DG2, Proposition D.11.4], if $\tilde{f} \in C_0^\infty\left(\left(0, \frac{b^2}{2}\right)\right)$ and $h \in C^\infty(\mathbb{R})$, supp $h \subset \left(\frac{b^2}{2} + \|\tilde{V}_0\|, \infty\right)$ and $h(s) = 1$ for $|s|$ large, then $h\left(\frac{p^2}{2}\right)\tilde{f}(\tilde{H})\langle x\rangle^N$ is bounded for any N. We choose such $\tilde{f}$ and h with $\tilde{f} = 1$ on supp f and $h\left(\frac{|\xi|^2}{2}\right) = 1$ for $\xi \in$ supp g (this requires $R \geq \sqrt{b^2 + 2\|\tilde{V}_0\|}$). Then for any N

$$\begin{aligned} \|g(p)\psi_t\| &= \left\| g(p)h\left(\frac{p^2}{2}\right)\tilde{f}(\tilde{H})\langle x\rangle^N\langle x\rangle^{-N}\psi_t \right\| \\ &\leq c_N\|\langle x\rangle^{-N}\psi_t\|, \end{aligned}$$

and the result follows from Proposition 5.2.

For the general case we use the result for $l = 0$, the fact that with $\tilde{f}$ given as above $\langle p\rangle^l g(p)\tilde{f}(\tilde{H})$ is bounded and various commutations. □

Remark. The proof in [DG2] shows that since $\tilde{V}_0 \geq 0$ we can take $R = b$.

In the following we choose a positive function $g \in C^\infty(\mathbb{R})$ with $g' \geq 0$, $g'' \geq 0$, $g(t) = t$ if $t \geq \frac{3}{4}$, and g constant for $t \leq \frac{1}{2}$. We set $\rho(x) = g(|x|)$, $\tilde{\omega} = \nabla\rho(x)$, and

$$p_{||} = \frac{1}{2}(p \cdot \tilde{\omega} + \tilde{\omega} \cdot p).$$

We will use the notation

$$\mathbf{D} = \frac{\partial}{\partial t} + i[\tilde{H}, \cdot].$$

Proposition 5.5 *Suppose $f \in C_0^\infty((0, L) \setminus \sigma_{pp}(\tilde{H}))$, $\theta < 0$, and l is a non-negative integer. Then*

$$\left\| \left(p_{||} - \frac{\rho(x)}{t}\right)^l \chi_{(-\infty,\theta)}\left(p_{||} - \frac{\rho(x)}{t}\right)\psi_t \right\| = \mathcal{O}(t^{-\infty}). \tag{5.7}$$

Proof. We introduce

$$A_b = B_2^* A B_2,\; A = tp_{||} - \rho(x),\; B_j = \chi_j(\frac{|x|}{t})f_j(\tilde{H}),$$

where with $f_0 = f$, $f_j(s) = 1$ on a neighborhood of the support of f_{j-1} for $j = 1, 2$, and similarly with $\chi_0 = \chi_{[\lambda_0,\Lambda_0]}$, $\chi_j(s) = 1$ on a neighborhood of the support of χ_{j-1}. Here λ_0 and Λ_0 are chosen in agreement with Propositions 5.2 and 5.3, and the functions f_1, f_2, χ_1 and χ_2 are smooth and compactly supported.

Clearly the localization operator in (5.7) is a (time-dependent) function of A. First we prove the bound with A replaced by A_b. Note that since $i\left[\tilde{V}_0, A\right] = 0$,

$$\mathbf{D}A_b = B_2^*\langle p, t\rho^{(2)}(x)p\rangle B_2 + B_2^* tk(x)B_2 + (B_2^* A\mathbf{D}B_2 + h.c.),$$

where $k \in C^\infty(\mathbb{R}^n)$ and $\partial_x^\alpha k(x) = \mathcal{O}(|x|^{-3-|\alpha|})$ for all α. Clearly the first term on the right hand side is non-negative. To invoke [Sk, Corollary 2.6] it suffices to show that for any $l, m \in \mathbb{N}$

$$\left\|\chi'(\frac{|x|}{t}) f_2(\tilde{H}) g_l(\frac{A_b}{t})\psi_t\right\| = \mathcal{O}(t^{-m}), \tag{5.8}$$

where χ' is bounded and $\chi'\chi_1 = 0$, and $(\frac{d}{ds})^k g(s) = O(\langle s\rangle^{l-k})$ for all $k \in \mathbb{N}$.

To show (5.8) we notice that for any $m \in \mathbb{N}$ and any semi-norm on $\mathcal{S}(\mathbb{R}^n)$

$$\left\|\psi_t - \chi_1^m(\frac{|x|}{t})\psi_t\right\|_\gamma = \mathcal{O}(t^{-\infty}), \tag{5.9}$$

cf. Propositions 5.2 – 5.4. Next we write $g_l(s) = g_{-1}(s)(s-i)^{l+1}$ and pick an almost analytic extension $\tilde{g}$ of g_{-1} satisfying

$$|\overline{\partial}\tilde{g}(z)| \le c_N \langle \text{Re } z\rangle^{-2-N} |\text{Im } z|^N, \quad N \ge 0,$$

$$\text{supp } \tilde{g} \subset \{z : \text{Im } z| \le c\langle \text{Re } z\rangle\},$$

cf. for example [DG2, Appendix C].

Using (5.9) it suffices to bound

$$\left\|\chi'(\frac{|x|}{t}) f_2(\tilde{H}) g_{-1}(\frac{A_b}{t})(\frac{A_b}{t} - i)^{l+1}\chi_1^m(\frac{|x|}{t})\psi_t\right\| = \mathcal{O}(t^{-m}),$$

and for this we first substitute the representation of $g_{-1}(A_b)$ in terms of the above extension, cf. the proof of Proposition 5.1. Then we move all of the m factors of $\chi_1(\frac{|x|}{t})$ one by one to the left. In each step we pick up the bound t^{-1} from all appearing commutations. Since $\chi'\chi_1 = 0$ all terms produced in each step involve a commutator. Since other conditions of [Sk, Corollary 2.6] are readily verified, (5.7) with A replaced by A_b follows from the conclusion of [Sk, Corollary 2.6].

We complete the proof of (5.7) by removing the localization factors B_2 in the bound obtained for A_b. Factorizing again $g_l(s) = g_{-1}(s)(s-i)^{l+1}$ and writing the resulting difference of products $g_l(A) - g_l(A_b)$ as a telescoping sum yields together with (5.9) (with χ_1 replaced by B_1) that it suffices to show that

$$\|T(t^{-1}A - i)^k B_1^m \psi_t\| = \mathcal{O}(t^{-m}),$$

where either

$$\begin{aligned} T &= g_{-1}(t^{-1}A) - g_{-1}(t^{-1}A_b) \\ &= \pi^{-1}\int \overline{\partial}\tilde{g}(z)(t^{-1}A - z)^{-1} t^{-1}(A_b - A)(t^{-1}A_b - z)^{-1}) d^2 z, \end{aligned}$$

or $T = t^{-1}(A - A_b)$. Noticing for both cases that $\|(A - A_b)B_1\| = \mathcal{O}(t^{-\infty})$ we may now proceed as above moving each of the factors of B_1 one by one to the left; in each step we pick up the bound t^{-1}. □

Proposition 5.6 *Suppose $f \in C_0^\infty((0,L)\setminus\sigma_{pp}(\tilde{H}))$. Then there is an $\epsilon > 0$ so that for all multi-indices α*

$$\left\| \left(\frac{x_\perp}{\langle x\rangle}\right)^\alpha \psi_t \right\| + \|(p_\perp)^\alpha \psi_t\| = \mathcal{O}(t^{-\epsilon|\alpha|}). \tag{5.10}$$

Proof. We consider the observable

$$Q = \frac{1}{2}(p^2 - p_{||}^2) + \tilde{V}_0(x) + \frac{\eta}{2}(p\cdot\nabla\tilde{V}_0(\tilde{\omega}) + \nabla\tilde{V}_0(\tilde{\omega})\cdot p),$$

where $\tilde{\omega} = \nabla\rho(x)$ ($= \omega$ if $|x| > \frac{3}{4}$). We will choose $\eta > 0$ and small so that for some small positive μ and ϵ *roughly*

$$Q \geq \mu\left(p^2 - p_{||}^2 + \frac{|x_\perp|^2}{|x|^2}\right),$$

and

$$\mathbf{D}Q = i[\tilde{H}, Q] \leq \frac{-2\epsilon}{t}Q.$$

This will lead to (5.10).

Let

$$p_\perp^x = p - \frac{p_{||}\tilde{\omega} + \tilde{\omega}p_{||}}{2}.$$

On vectors supported in $|x| \geq 1$ and for any $\gamma > 0$, we have

$$\begin{aligned}
\pm(p\cdot\nabla\tilde{V}_0(\tilde{\omega}) + \nabla\tilde{V}_0(\tilde{\omega})\cdot p) &= \pm(p_\perp^x\cdot\nabla\tilde{V}_0(\omega) + \nabla\tilde{V}_0(\omega)\cdot p_\perp^x) \\
&= -\gamma(p_\perp^x)^2 - \gamma^{-1}|\nabla\tilde{V}_0(\omega)|^2 \\
&\quad + (\sqrt{\gamma}\,p_\perp^x \pm \sqrt{\gamma^{-1}}\nabla\tilde{V}_0(\omega))^2.
\end{aligned} \tag{5.11}$$

Again on vectors supported in $|x| \geq 1$,

$$|p_\perp^x|^2 = p^2 - p_{||}^2,$$

and thus setting $\gamma = (2\eta)^{-1}$ in (5.11) we obtain

$$Q = \frac{p^2 - p_{||}^2}{4} + \tilde{V}_0(\omega) - \eta^2|\nabla\tilde{V}_0(\omega)|^2 + \frac{\eta}{2}(\sqrt{\gamma}p_\perp^x + \sqrt{\gamma^{-1}}\nabla\tilde{V}_0(\omega))^2, \tag{5.12}$$

$$Q = \frac{3}{4}(p^2 - p_{||}^2) + \tilde{V}_0(\omega) + \eta^2|\nabla\tilde{V}_0(\omega)|^2 - \frac{\eta}{2}(\sqrt{\gamma}p_\perp^x - \sqrt{\gamma^{-1}}\nabla\tilde{V}_0(\omega))^2, \tag{5.13}$$

on vectors supported in $|x| \geq 1$. For some $c > 0$,

$$c^{-1}\left(\frac{|x_\perp|}{|x|}\right)^2 \leq \tilde{V}_0(\omega),\ |\nabla\tilde{V}_0(\omega)|^2 \leq c\left(\frac{|x_\perp|}{|x|}\right)^2,$$

so that if $\eta > 0$ and small enough

$$\tilde{V}_0(\omega) - \eta^2 |\nabla \tilde{V}_0(\omega)|^2 \geq \mu \left(\frac{|x_\perp|}{|x|} \right)^2$$

for all $\mu \leq \mu_1$ independent of η. Taking $\mu \in \left(0, \mathrm{Min}\left(\frac{1}{4}, \mu_1\right)\right)$ we consequently obtain from (5.12)

$$Q \geq \mu \left(p^2 - p_{\|}^2 + \left(\frac{|x_\perp|}{|x|} \right)^2 \right), \tag{5.14}$$

as quadratic forms on vectors supported in $|x| \geq 1$.

Again on vectors supported in $|x| \geq 1$ we calculate

$$\begin{aligned} \mathbf{D}Q &= -\langle pMr^{-\frac{1}{2}}, p_{\|} r^{-\frac{1}{2}} Mp \rangle - \frac{\eta |\nabla \tilde{V}_0(\omega)|^2}{r} \\ &+ \frac{\eta}{2} \, \mathrm{Re} \sum_{j,k,\ell} p_j \left(p_k \frac{M_{k\ell}}{r} \tilde{V}^{(2)}_{0\ell j}(\omega) + \tilde{V}^{(2)}_{0j\ell}(\omega) \frac{M_{\ell k}}{r} p_k \right) + g_1(x), \end{aligned}$$

where $r = |x|$,

$$M_{k\ell} = \delta_{k\ell} - \omega_k \omega_\ell;$$
$$\tilde{V}^{(2)}_{0\ell j}(x) = \partial_\ell \partial_j \tilde{V}_0(x),$$

and g_1 is a symbol of order -3, i.e., $|\partial_x^\alpha g_1(x)| \leq c_\alpha \langle x \rangle^{-3-|\alpha|}$ for all α. We continue the calculation using

$$\tilde{V}_0^{(2)}(\omega) = \tilde{V}_0^{(2)}(\omega) M - \langle \omega, \cdot \rangle \nabla \tilde{V}_0(\omega),$$

which follows from the homogeneity of $\tilde{V}_0$ (for $|x| > \frac{1}{2}$), obtaining

$$\begin{aligned} \mathbf{D}Q &= -\langle pMr^{-\frac{1}{2}}, (p_{\|} - \eta \tilde{V}_0^{(2)}(\omega)) r^{-\frac{1}{2}} Mp \rangle \\ &- \eta \, \mathrm{Re} \, \langle pMr^{-\frac{1}{2}} \omega \cdot p, r^{-\frac{1}{2}} \nabla \tilde{V}_0(\omega) \rangle \\ &- \frac{\eta |\nabla \tilde{V}_0(\omega)|^2}{r} + g_2(x), \end{aligned}$$

with g_2 a symbol of order -3. Using

$$-2 \, \mathrm{Re} \, A^* B = -(A+B)^*(A+B) + A^*A + B^*B,$$

we obtain

$$\begin{aligned} \mathbf{D}Q &= -\left\langle pMr^{-\frac{1}{2}}, \left(p_{\|} - \eta \tilde{V}_0^{(2)}(\omega) - \frac{1}{2} \eta |p \cdot \omega|^2 \right) r^{-\frac{1}{2}} Mp \right\rangle \\ &- \frac{1}{2} \frac{\eta |\nabla \tilde{V}_0(\omega)|^2}{r} - \frac{1}{2} \sum_j |p \cdot \omega r^{-\frac{1}{2}} (Mp)_j + r^{-\frac{1}{2}} \partial_j \tilde{V}_0(\omega)|^2 + g_2(x), \end{aligned} \tag{5.15}$$

where we are using the notation $|B|^2 = B^*B$. Note for later use that $|p \cdot \omega|^2 = |\omega \cdot p|^2 + \frac{n-1}{r^2}$. All calculations are valid on vectors with support in $|x| \geq 1$.

Let
$$z^\alpha = z_{i_1}^{\alpha_1} z_{i_2}^{\alpha_2} \cdots z_{i_k}^{\alpha_k}, \quad z_j = p_j \text{ or } \frac{x_j}{\langle x \rangle},$$
where we always take $i_1, \ldots, i_k \geq 2$. Assume inductively that for some $m \geq 1$ and all $|\alpha| \leq m-1$ that
$$\|z^\alpha \psi_t\| = \mathcal{O}(t^{-\epsilon|\alpha|}), \tag{5.16}$$
where ϵ will be given later. We calculate
$$\frac{d}{dt}(\psi_t, Q^m \psi_t) = \sum_{k=0}^{m-1} \left(\psi_t, Q^k \mathbf{D} Q Q^{m-(k+1)} \psi_t\right). \tag{5.17}$$

Because of Propositions 5.2 and 5.4 we can replace ψ_t by $\chi\psi_t$ where $\chi(x) = 0$ for $|x| \leq 1$, $0 \leq \chi \leq 1$, and $1 - \chi \in C_0^\infty(\mathbb{R}^n)$, making an error of $\mathcal{O}(t^{-\infty})$ in (5.28). If m is odd, we can then commute $\mathbf{D}Q$ through as many factors of Q as necessary to obtain
$$\begin{aligned}\frac{d}{dt}(\psi_t, Q^m \psi_t) &= m\left(\chi\psi_t, Q^{\frac{m-1}{2}} \mathbf{D} Q Q^{\frac{m-1}{2}} \chi\psi_t\right) \\ &\quad + \text{commutator terms} + \mathcal{O}(t^{-\infty}).\end{aligned} \tag{5.18}$$

The commutator terms can be written
$$\sum_{\ell_1+\ell_2+\ell_3=m-1} \left(\chi\psi_t, Q^{\ell_1} ad_Q^{\ell_2}(\mathbf{D}Q)\, Q^{\ell_3} \chi\psi_t\right) c_{\ell_1\ell_2\ell_3}, \tag{5.19}$$
where the coefficients $c_{\ell_1\ell_2\ell_3}$ are combinatoric factors. We can arrange things so that $c_{\ell_1\ell_2\ell_3} = 0$ unless $|\ell_1 - \ell_3| \leq 1$ and $\ell_2 \geq 2$. The latter can be seen from reality considerations. We now use the fact that each commutator introduces another factor of r^{-1} which in conjunction with Propositions 5.2 and 5.4 implies that (5.19) can be bounded by
$$c \sum_{\ell_1+\ell_2+\ell_3=m-1} |c_{\ell_1\ell_2\ell_3}| t^{-(\ell_2+1)} \|Q^{\ell_1}\chi\psi_t\| \cdot \|Q^{\ell_3}\chi\psi_t\| + \mathcal{O}(t^{-\infty}). \tag{5.20}$$

On vectors supported in $|x| \geq 1$, we have
$$(p_\perp^x)_1 = \frac{1}{2}\left[(p_1 + p_{||})\left(1 - \frac{x_1}{r}\right) + \left(1 - \frac{x_1}{r}\right)(p_1 + p_{||})\right] - \frac{1}{2}\left(\frac{x_\perp}{r} \cdot p_\perp + p_\perp \cdot \frac{x_\perp}{r}\right), \tag{5.21}$$
and for $j \geq 2$,
$$(p_\perp^x)_j = p_j - \left(p_{||}\left(\frac{x_j}{r}\right) + \left(\frac{x_j}{r}\right) p_{||}\right)/2. \tag{5.22}$$

Since if $c_{\ell_1\ell_2\ell_3} \neq 0$, $2\ell_1 \leq m-1$ and $2\ell_3 \leq m-1$, we can use the induction hypothesis along with the definition of Q, the first equality of (5.11), the equality $|p_\perp^x|^2 = p^2 - p_{||}^2$

previously mentioned, and (5.21) and (5.22) along with Propositions 5.2 and 5.4 to conclude that

$$\|Q^{\ell_j}\chi\psi_t\| = \mathcal{O}(t^{-2\epsilon\ell_j}), \quad j = 1, 3.$$

Thus (5.20) can be bounded by

$$c' \sum_{\ell_1+\ell_2+\ell_3=m-1} |c_{\ell_1\ell_2\ell_3}| t^{-2\epsilon(\ell_1+\ell_2+\ell_3+1)} t^{-(\ell_2+1)(1-2\epsilon)}. \tag{5.23}$$

We demand $(\ell_2+1)(1-2\epsilon) > 1$. Since we know $\ell_2 \geq 2$, this means we must take $\epsilon < \frac{1}{3}$, in which case (5.23) is bounded by

$$c'' t^{-2\epsilon m} t^{-1-\delta}$$

for some $\delta > 0$.

We now consider the first term on the right side of (5.18) and use (5.15). In bounding this term from above, according to Proposition 5.4, we can replace $|p\cdot\omega|^2$ with $\Lambda^2 + \frac{n-1}{r^2}$ for large enough Λ up to an error of $\mathcal{O}(t^{-\infty})$. The operator $p_{\|}$ can be written

$$\begin{aligned} p_{\|} &= p_{\|} - \frac{\rho(x)}{t} + \frac{\rho(x)}{t} \\ &= \left(p_{\|} - \frac{\rho(x)}{t} + \theta\right) \tilde{\chi}_\theta \left(p_{\|} - \frac{\rho(x)}{t}\right) \\ &+ \left(p_{\|} - \frac{\rho(x)}{t} + \theta\right)\left(1 - \tilde{\chi}_\theta\left(p_{\|} - \frac{\rho(x)}{t}\right)\right) + \frac{\rho(x)}{t} - \theta. \end{aligned} \tag{5.24}$$

Here $\theta > 0$, $\tilde{\chi}_\theta \in C^\infty(\mathbb{R})$ satisfies $0 \leq \tilde{\chi}_\theta \leq 1$, $\tilde{\chi}_\theta(s) = 1$, if $s \leq -\theta$, and $\tilde{\chi}_\theta(s) = 0$ if $s \geq -\frac{\theta}{2}$. The second term on the right side of (5.24) is non-negative, and we claim that the first term contributes $\mathcal{O}(t^{-\infty})$ to (5.18). To see this, let $F(s) = (s+\theta)\tilde{\chi}_\theta(s)$, $B(t) = p_{\|} - \frac{\rho(x)}{t}$, $A = (r^{-\frac{1}{2}}Mp)_j Q^{\frac{m-1}{2}}\chi$, and ℓ a large positive integer. For a suitable almost analytic extension $\tilde{F}$ of F we easily derive

$$\begin{aligned} F(B(t))A &= \sum_{j=0}^{\ell} ad^j_{B(t)}(A) F^{(j)}(B(t)) \\ &+ \frac{(-1)^\ell}{\pi}\int \overline{\partial}\tilde{F}(z)(B(t)-z)^{-1} ad^{\ell+1}_{B(t)}(A)(B(t)-z)^{-(\ell+1)} d^2 z. \end{aligned} \tag{5.25}$$

The sum on the right side of (5.25) contributes $\mathcal{O}(t^{-\infty})$ to (5.18) because of Proposition 5.5, while the last term contributes $\mathcal{O}(t^{-(\ell+3/2)})$. Since ℓ is arbitrarily large, this establishes our claim. We thus obtain

$$\begin{aligned} \frac{d}{dt}(\psi_t, Q^m\psi_t) \;&\leq\; -m\left(Q^{\frac{m-1}{2}}\chi\psi_t, \left(\left\langle pM, r^{-1}\left(t^{-1}|x| - \theta - \eta\tilde{V}_0^{(2)}(\omega) - \frac{\eta}{2}\Lambda^2\right)Mp\right\rangle\right.\right. \\ &+ \left.\left.\frac{\eta}{2}|\nabla\tilde{V}_0(\omega)|^2\right) Q^{\frac{m-1}{2}}\chi\psi_t\right) + \mathcal{O}(t^{-2\epsilon m}t^{-1-\delta}). \end{aligned} \tag{5.26}$$

Let $\tilde{\chi}_{\lambda_0,\Lambda_0} \in C^\infty(\mathbb{R})$ with $0 \le \tilde{\chi}_{\lambda_0,\Lambda_0} \le 1$, $\tilde{\chi}_{\lambda_0,\Lambda_0}(s) = 0$, if $s < \lambda_0$ or $s > \Lambda_0$ and $\tilde{\chi}_{\lambda_0,\Lambda_0}(s) = 1$, if $\lambda_0 + \delta_1 \le s \le \Lambda_0 - \delta_1$ for some small $\delta_1 > 0$. We insert $\tilde{\chi}_{\lambda_0,\Lambda_0}\left(\frac{|x|}{t}\right)$ in front of $\left(t^{-1}|x| - \theta - \eta\tilde{V}_0^{(2)}(\omega) - \frac{\eta}{2}\Lambda^2\right)/r$ and $|\nabla\tilde{V}_0(\omega)|^2/r$. If λ_0 is sufficiently small (but > 0), $\delta_1 = 2^{-1}\lambda_0$ and Λ_0 sufficiently large, the error from this contributes $\mathcal{O}(t^{-\infty})$ to the right side of (5.26) because of Propositions 5.2 and 5.3. If η and θ are chosen small enough we obtain

$$\begin{aligned}\frac{d}{dt}(\psi_t, Q^m\psi_t) \;\le\; & -\frac{m}{t}\left(Q^{\frac{m-1}{2}}\chi\psi_t, \left(\frac{\lambda_0}{2\Lambda_0}\langle p, Mp\rangle + \frac{\eta}{2\Lambda_0}|\nabla\tilde{V}_0(\omega)|^2\right) Q^{\frac{m-1}{2}}\chi\psi_t\right) \\ & + \mathcal{O}(t^{-2\epsilon m}t^{-1-\delta}).\end{aligned}$$

We use $\langle p, Mp\rangle = |p_\perp^x|^2 + \frac{c_1}{r^2}$ and the fact that $\tilde{V}_0(\omega) \le c_2|\nabla\tilde{V}_0(\omega)|^2$ if $\frac{x_1}{r} \ge 0$, in conjunction with Proposition 5.1 to deal with the region $\frac{x_1}{r} < 0$, and we obtain

$$\begin{aligned}\frac{d}{dt}(\psi_t, Q^m\psi_t) \;\le\; & -\frac{m}{t}\Bigg(Q^{\frac{m-1}{2}}\chi\psi_t, \Bigg(\frac{\lambda_0}{2\Lambda_0}(p^2 - p_{\|}^2) \\ & + \frac{\eta}{2\Lambda_0}(c_2+\eta^2)^{-1}(\tilde{V}_0(\omega) + \eta|\nabla\tilde{V}_0(\omega)|^2)\Bigg) Q^{\frac{m-1}{2}}\chi\psi_t\Bigg) \\ & + \mathcal{O}(t^{-2\epsilon m}t^{-1-\delta}).\end{aligned}$$

Using (5.13) we see that if $\eta > 0$ is small enough

$$\frac{d}{dt}(\psi_t, Q^m\psi_t) \le -\frac{2m}{t}\left(\frac{\eta}{8\Lambda_0 c_2}\right)(\chi\psi_t, Q^m\chi\psi_t) + \mathcal{O}(t^{-2m\epsilon}t^{-1-\delta}),$$

so if, in addition, we make sure that $\frac{\eta}{8\Lambda_0 c_2} < \frac{1}{3}$, we can define

$$\epsilon = \frac{\eta}{8\Lambda_0 c_2}.$$

Then another application of Proposition 5.2 gives

$$\frac{d}{dt}(\psi_t, Q^m\psi_t) \le -\frac{2m\epsilon}{t}(\psi_t, Q^m\psi_t) + \mathcal{O}(t^{-2m\epsilon}t^{-1-\delta}), \tag{5.27}$$

which upon integration implies

$$(\psi_t, Q^m\psi_t) = \mathcal{O}(t^{-2m\epsilon}). \tag{5.28}$$

We now use the Morse lemma and (5.12) to write Q as a sum of squares of self-adjoint operators (on vectors with support in $|x| \ge 1$). This is already almost the case in (5.12) since $(p^x_\perp)^2 = \sum_{j=1}^n (p^x_\perp)^2_j$, but the quantity $\tilde{V}_0(\omega) - \eta^2|\nabla \tilde{V}_0(\omega)|^2$ needs some work. For small η fixed, we write

$$\tilde{V}_0(\omega) - \eta^2|\nabla \tilde{V}_0(\omega)|^2 = \sum_{j=2}^n u_j^2$$

in a neighborhood of e_1 where u_j is C^∞ in this neighborhood and has an expression

$$u_j = \sum_{k=2}^n a_{jk}(\omega)\frac{x_k}{r}$$

(see [Mil]) with a_{jk} also C^∞. Since for small η, $\tilde{V}_0(\omega) - \alpha^2|\nabla \tilde{V}_0(\omega)|^2$ is positive away from e_1, it has a C^∞ square root, u_1. Thus using a partition of unity, $\chi_1^2(\omega)+\chi_2^2(\omega) = 1$, for S^{n-1}, with $\chi_1(\omega) = 1$ in a small neighborhood of e_1 and zero outside a slightly larger neighborhood, we obtain

$$\tilde{V}_0(\omega) - \eta^2|\nabla \tilde{V}_0(\omega)|^2 = \sum_{j=2}^n (\chi_1(\omega)u_j)^2 + (\chi_2(\omega)u_j)^2.$$

Notice that $|\chi_2(\omega)u_j| \leq c\frac{|x_\perp|}{r}$ except in a small neighborhood of $-e_1$. We will later use Proposition 5.1 to bound contributions from this neighborhood. We write

$$Q = \sum_{k=1}^J A_k^2,$$

and thus

$$\begin{aligned}\chi Q^m \chi &= \sum_{|\alpha|=m} \frac{m!}{\alpha_1!\cdots\alpha_J!}\chi A_1^{\alpha_1}\cdots A_J^{\alpha_J} A_J^{\alpha_J}\cdots A_1^{\alpha_1}\chi \\ &\quad + \sum_{|\gamma|+|\beta|\leq 2m-2} \chi A_1^{\gamma_1}\cdots A_J^{\gamma_J} C_{\gamma\beta} A_J^{\beta_J}\cdots A_1^{\beta_1}\chi.\end{aligned}$$

Here $C_{\gamma\beta}$ is a linear combination of products of multiple commutators of the A_J's. We can arrange $|\beta| \leq m-1$, $|\gamma| \leq m-1$ in this sum. $C_{\gamma\beta}$ involves $k = 2m - (|\gamma|+|\beta|)$ A_j's which in a worst-case scenario contribute $\mathcal{O}(t^{-k/2})$ to $(\psi_t, \chi Q^m \chi\psi_t)$. Using the induction hypothesis and removing χ from the left side, we obtain

$$\begin{aligned}(\psi_t, Q^m\psi_t) &\geq \sum_{|\alpha|=m} \frac{m!}{\alpha_1!\cdots\alpha_2!\cdots\alpha_J!}\|A_J^{\alpha_J}\cdots A_1^{\alpha_1}\chi\psi_t\|^2 \\ &\quad -c\sum_{|\gamma|+|\beta|\leq 2m-2} t^{-\epsilon(|\gamma|+|\beta|)}t^{-\frac{k}{2}}. \qquad (5.29)\end{aligned}$$

Using (5.28), (5.29) implies

$$\|A_J^{\alpha_J}\cdots A_1^{\alpha_1}\chi\psi_t\| = \mathcal{O}(t^{-m\epsilon}), \tag{5.30}$$

as long as $\epsilon \leq \frac{1}{2}$ (but we have already required $\epsilon < \frac{1}{3}$). Using the induction hypothesis and Proposition 5.1, we obtain

$$\|A_J^{\alpha_J}\cdots A_1^{\alpha_1}\chi\psi_t\| = \mathcal{O}(t^{-\epsilon|\alpha|}), \tag{5.31}$$

for all $|\alpha| \leq m$. We now write each $\frac{x_j}{r}$, $j \geq 2$, as a linear combination of the A_j's with coefficients which are symbols of order zero:

$$\frac{x_j}{r} = \sum_{k=2}^{n} h_{jk}(\omega)\chi_1(\omega)u_k + h_{j1}(\omega)\chi_2(\omega)u_1,$$

and similarly for p_j, $j \geq 2$ (see (5.22)). Using this and (5.31), we obtain (5.16) for $|\alpha| \leq m$. This completes the induction when m is odd.

If m is even, (5.18) is not correct. We obtain instead (for $m \geq 2$),

$$\begin{aligned} \frac{d}{dt}(\psi_t, Q^m\psi_t) \;&=\; m\ \mathrm{Re}\left(Q^{\frac{m}{2}-1}\chi\psi_t, (\mathbf{D}Q\,Q)\,Q^{\frac{m}{2}-1}\chi\psi_t\right) \\ &\quad + {\textstyle\sum}' c_k\ \mathrm{Re}\left(Q^k\chi\psi_t, ad_Q^{2\ell}(\mathbf{D}Q)\,Q^{k+1}\chi\psi_t\right) \\ &\quad + \mathcal{O}(t^{-\infty}). \end{aligned} \tag{5.32}$$

In the sum above, $k + \ell + 1 = \frac{m}{2}$ and $\ell \geq 1$. Using $Q = \sum_j A_j^2$ as above, we obtain

$$\frac{1}{2}\left(Q\,\mathbf{D}Q + \mathbf{D}Q\,Q\right) = \sum_j A_j \mathbf{D}Q\,A_j + \frac{1}{2}\sum_j \left[A_j, [A_j, \mathbf{D}Q]\right], \tag{5.33}$$

and estimate as before. At a later stage we need to put the A_j's in their rightful place. The term

$$-\frac{2\epsilon}{t}\sum_j A_j Q A_j$$

is encountered and is replaced by

$$-\frac{2\epsilon}{t}Q^2 + \frac{\epsilon}{t}\sum_j [A_j, [A_j, Q]] \tag{5.34}$$

using an identity similar to (5.33). The commutator terms in (5.33) and (5.34) are easy to estimate as above leading to

$$(\psi_t, Q^m\psi_t) = \mathcal{O}(t^{-2m\epsilon}).$$

The remainder of the proof goes through without change from the case $m =$ odd. $\square$

Proposition 5.7 *Suppose $f \in C_0^\infty((0,L) \setminus \sigma_{pp}(\tilde{H}))$ and that $\left\| \left(\frac{x_\perp}{|x|}\right)^\alpha \psi_t \right\| = \mathcal{O}(t^{-\epsilon_0 |\alpha|})$ for all α and some $\epsilon_0 > 0$. Then if $\epsilon_1 < \mathrm{Min}\{2\epsilon_0, 1\}$,*

$$\left\| \left(p_1 - \frac{x_1}{t}\right)^m \psi_t \right\| = \mathcal{O}(t^{-m\epsilon_1}) \tag{5.35}$$

for all $m \geq 0$.

Proof. Let $B_1(t) = p_1 - \frac{x_1}{t}$ and compute

$$\mathbf{D}B_1(t) = -t^{-1}B_1(t) - \partial_1 \tilde{V}_0(x).$$

Thus

$$\begin{aligned} \mathbf{D}B_1(t)^{2m} &= -\frac{2m}{t}B_1(t)^{2m} - mB^{m-1}(\partial_1\tilde{V}_0 B_1(t) + B_1(t)\partial_1\tilde{V}_0)B_1(t)^{m-1} \\ &\quad - \mathrm{Re}\sum_{j=0}^{m-2} c_j B_1(t)^j((i\partial_1)^k \partial_1 \tilde{V}_0)B_1(t)^{2m-(j+k+1)}, \end{aligned} \tag{5.36}$$

where in (5.36) the c_j are integers, $2m-(j+k+1) = j+1$, $k = 2(m-j-1) \geq 2$. We make the induction hypothesis

$$\|B_1(t)^\ell \psi_t\| = \mathcal{O}(t^{-\ell\epsilon_1}), \tag{5.37}$$

for all $\ell \leq m-1$. It follows from the hypothesis that for any $\theta \in (0, \epsilon_0)$,

$$\left\| \chi_{[1,\infty)}\left(t^{\epsilon_0-\theta}\frac{|x_\perp|}{|x|}\right)\psi_t \right\| = \mathcal{O}(t^{-\infty}), \tag{5.38}$$

and since for $|x| \geq \frac{1}{2}$, $\partial_1 \tilde{V}_0(x) = -\frac{1}{x_1}x_\perp \cdot \nabla_\perp \tilde{V}_0(x)$, we have for $\frac{x}{|x|}$ in a neighborhood of e_1,

$$|(i\partial_1)^k \partial_1 \tilde{V}_0(x)| \leq c_k \left(\frac{|x_\perp|}{|x|}\right)^2 |x|^{-k-1}. \tag{5.39}$$

Combining (5.38), (5.39), Propositions 5.1, 5.2, 5.3, 5.4 and the induction hypothesis (5.37), we obtain

$$\begin{aligned} \mathrm{Re}\left(\psi_t, B_1(t)^j[(i\partial_1)^k(\partial_1\tilde{V}_0)]B_1(t)^{j+1}\psi_t\right) &= \mathcal{O}(t^{-(2j+1)\epsilon_1}t^{-2(\epsilon_0-\theta)}t^{-k-1}) \\ &= \mathcal{O}\left(t^{-(2m-(k+1))\epsilon_1-(k+1)-2(\epsilon_0-\theta)}\right). \end{aligned} \tag{5.40}$$

Since $k \geq 2$, and $\epsilon_1 < \mathrm{Min}\{2\epsilon_0, 1\}$, we have

$$(2m-(k+1))\epsilon_1 + k + 1 + 2(\epsilon_0 - \theta) \geq (2m+1)\epsilon_1 + (2\epsilon_0 - \epsilon_1 - 2\theta).$$

We choose $\theta \in \left(0, \frac{2\epsilon_0-\epsilon_1}{2}\right)$, and let $\delta = 2\epsilon_0 - \epsilon_1 - 2\theta$. Then (5.40) is $\theta(t^{-2m\epsilon_1}t^{-1-\delta})$. Thus

$$\begin{aligned}\frac{d}{dt}(\psi_t, B_1(t)^{2m}\psi_t) &= -2mt^{-1}(\psi_t, B_1(t)^{2m}\psi_t) \\ &\quad - 2m \operatorname{Re}(B_1(t)^{m-1}\psi_t, \partial_1\tilde{V}_0 B_1(t)B_1(t)^{m-1}\psi_t) \\ &\quad + \mathcal{O}(t^{-2m\epsilon_1}t^{-1-\delta}). \end{aligned} \tag{5.41}$$

We have

$$-2 \operatorname{Re}[(\partial_1\tilde{V}_0)B_1(t)] \leq \frac{\gamma}{t}B_1(t)^2 + \gamma^{-1}t(\partial_1\tilde{V}_0)^2,$$

so that estimating the term involving $(\partial_1\tilde{V}_0)^2$ as before, we have

$$\begin{aligned}\frac{d}{dt}(\psi_t, B_1(t)^{2m}\psi_t) &= -\frac{2m}{t}\left(1-\frac{\gamma}{2}\right)(\psi_t, B_1(t)^{2m}\psi_t) \\ &\quad + \mathcal{O}(t^{-2m\epsilon_1}t^{-1-\delta}). \end{aligned} \tag{5.42}$$

If we choose γ suitably small, integrating (5.42) gives

$$(\psi_t, B_1(t)^{2m}\psi_t) = \mathcal{O}(t^{-2m\epsilon_1}). \qquad \Box$$

Proposition 5.8 *Suppose $f \in C_0^\infty((0,L) \setminus \sigma_{pp}(\tilde{H}))$ and suppose $h \in C^\infty(\mathbb{R})$ with $h' \in C_0^\infty(\mathbb{R})$ and $\left\{s : s > 0, \frac{s^2}{2} \in \operatorname{supp} f\right\} \cap \operatorname{supp} h = \emptyset$. Then*

$$\left\| h\left(\frac{x_1}{t}\right)\psi_t \right\| = \mathcal{O}(t^{-\infty}). \tag{5.43}$$

Proof. Since supp f is compact, we can find $k \in C_0^\infty((0,\infty))$ such that $k(s) = 1$ in a neighborhood of $\left\{s : s > 0, \frac{s^2}{2} \in \operatorname{supp} f\right\}$ but supp $k \cap$ supp $h = \emptyset$. We will show

$$\left\| h\left(\frac{x_1}{t}\right)k(p_1)\psi_t \right\| = \mathcal{O}(t^{-\infty}), \tag{5.44}$$

and

$$\|(1-k(p_1))\psi_t\| = \mathcal{O}(t^{-\infty}). \tag{5.45}$$

For each $s \in$ supp k there is an open interval I_s containing s with $\overline{I_s} \cap$ supp $h = \emptyset$. By compactness there is a finite number of the I_s say, $I_1, \ldots, I_m$ which cover supp k. We find a partition of unity subordinate to this cover, $\chi_1, \ldots, \chi_m$ with $\chi_j \in C_0^\infty$. For each j we can find $\chi_{j1}^\#$, $\chi_{j2}^\#$ with supp $\chi_{j1}^\#$ to the right of $\overline{I_j}$ and supp $\chi_{j2}^\#$ to the left of $\overline{I_j}$ such that ${\chi_{j1}^\#}'$, ${\chi_{j2}^\#}' \in C_0^\infty(\mathbb{R})$ and $(\chi_{j1}^\# + \chi_{j2}^\#)h = h$. Then

$$h\left(\frac{x_1}{t}\right)k(p_1) = \sum_{j,k}\chi_{jk}^\#\left(\frac{x_1}{t}\right)h\left(\frac{x_1}{t}\right)\chi_j(p_1)k(p_1).$$

By treating each term in the sum individually we can assume in proving (5.44) that either supp h is to the left or to the right of supp k. Consider the case where supp h is to the right of supp k. Then we can write $h\left(\frac{x_1}{t}\right) = \chi_3\left(\frac{x_1}{t} - \epsilon_3\right)$ with $\chi_3' \in C_0^\infty(\mathbb{R})$ with supp $\chi_3 \subset (0,\infty)$ and with ϵ_3 just to the left of supp h. Similarly we can write $k(p_1) = \chi_2(-p_1 - \epsilon_2)$ where $\chi_2 \in C_0^\infty((0,\infty))$ and $-\epsilon_2$ is just to the right of supp k. We can thus arrange $\epsilon_3 + \epsilon_2 > 0$. We choose χ_1 with $\chi_1' \in C_0^\infty(\mathbb{R})$ and supp $\chi_1 \subset (0,\infty)$. In fact, we choose $0 < \epsilon_1 < \epsilon_2 + \epsilon_3$ and $\chi_1(s) = 0$ if $s \leq \frac{\epsilon_1}{4}$, $\chi_1(s) = 1$ if $s \geq \frac{\epsilon_1}{2}$. Referring to Lemma 4.3 and performing the unitary transformation $U = S_{t^{-1}}\mathcal{F}^{-1}e^{itp_1^2/2}$ with $\mathcal{F}$ the Fourier transform and $S_{t^{-1}}$ the unitary scale transformation $x_1 \to \frac{x_1}{t}$ we obtain after taking the adjoint that

$$\begin{aligned}
&\left\|\chi_1\left(\epsilon_1 - \frac{x_1}{t}\right)\chi_2\left(\frac{x_1}{t} - p_1 - \epsilon_2\right)\chi_3(p_1 - \epsilon_3)\right\| \\
&= \left\|\chi_3\left(\frac{x_1}{t} - \epsilon_3\right)\chi_2(-p_1 - \epsilon_2)\chi_1\left(\epsilon_1 + p_1 - \frac{x_1}{t}\right)\right\|. \qquad (5.46)
\end{aligned}$$

Thus

$$\left\|h\left(\frac{x_1}{t}\right)k(p_1)\chi_1\left(\epsilon_1 + p_1 - \frac{x_1}{t}\right)\right\| = \mathcal{O}(t^{-\infty}),$$

while

$$\left\|\left(1 - \chi_1\left(\epsilon_1 + p_1 - \frac{x_1}{t}\right)\right)\psi_t\right\| = \mathcal{O}(t^{-\infty}),$$

by Propositions 5.6 and 5.7. If supp h is to the left of supp k a similar argument after applying the unitary reflection $x_1 \to -x_1$ gives the same result. This proves (5.44).

We now prove (5.45) where $k \in C_0^\infty((0,\infty))$ and $k(s) = 1$ in a neighborhood of $\{s : s > 0, \frac{s^2}{2} \in \text{supp } f\}$. We first claim that we can replace $1 - k(p_1)$ by $1 - (k(p_1) + k(-p_1))$ in (5.45) only making an error of $\mathcal{O}(t^{-\infty})$. This follows by the same reasoning as above, for if ϵ_3 is an arbitrarily small positive number and $\chi_3 \in C^\infty$ with supp $\chi_3 \subset (0,\infty)$, $\chi_3' \in C_0^\infty$,

$$\chi_3(-p_1 - \epsilon_3)\psi_t = \chi_3(-p_1 - \epsilon_3)\chi_2\left(\frac{x_1}{t} - \epsilon_2\right)\psi_t + \mathcal{O}(t^{-\infty}),$$

if χ_2 and $\epsilon_2 > 0$ are chosen suitably with supp $\chi_2 \subset (0,\infty)$, $\chi_2' \in C_0^\infty(\mathbb{R})$. This follows by Propositions 5.1 and 5.2. Then a suitable choice of ϵ_1 with $0 < \epsilon_1 < \epsilon_2 + \epsilon_3$ and χ_1 with supp $\chi_1 \subset (0,\infty)$ gives

$$\left\|\chi_3(-p_1 - \epsilon_3)\chi_2\left(\frac{x_1}{t} - \epsilon_2\right)\chi_1\left(\epsilon_1 + p_1 - \frac{x_1}{t}\right)\right\| = \mathcal{O}(t^{-\infty}), \qquad (5.47)$$

while

$$\left\|\left(1 - \chi_1\left(\epsilon_1 + p_1 - \frac{x_1}{t}\right)\right)\psi_t\right\| = \mathcal{O}(t^{-\infty}),$$

by Propositions 5.6 and 5.7. (5.47) follows from Lemma 4.3 after performing another unitary transformation implementing $x_1 \to tp_1$, $p_1 \to -\frac{x_1}{t}$ followed by the anti-unitary complex conjugation. The equation $k(p_1) + k(-p_1) = k_1\left(\frac{p_1^2}{2}\right)$ defines $k_1 \in C_0^\infty(\mathbb{R})$ with $k_1 = 1$ in a neighborhood of supp f.

Let $g = 1 - k_1$ and choose G smooth and $= 1$ in a neighborhood of supp g, but $Gf = 0$. Thus $1 - G \in C_0^\infty(\mathbb{R})$. Let $\delta > 0$ and choose $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(x) = 1$ if $|x| < 1$ and $\chi(x) = 0$ if $|x| \geq 2$. Define $\chi_\delta(x) = \chi\left(\frac{x}{\delta}\right)$. Then clearly,

$$W_\delta \equiv \frac{1}{2}|p_\perp|^2\chi_\delta(p_\perp) + \tilde{V}_0(x)\chi_\delta\left(\frac{x_\perp}{|x|}\right)\chi(2x)$$

satisfies

$$\lim_{\delta\downarrow 0}\|W_\delta\| = 0.$$

Choose $\delta > 0$ so that $\mathrm{dist}(\mathrm{supp}\, g, \mathrm{supp}(1-G)) > \|W_\delta\|$. Then according to [DG2, Proposition D.11.4],

$$\left\|g\left(\frac{p_1^2}{2}\right)\left(1 - G\left(\frac{p_1^2}{2} + W_\delta\right)\right)\langle x\rangle^N\right\| < \infty$$

for any N. From Proposition 5.2, it follows that

$$\left\|g\left(\frac{p_1^2}{2}\right)\left(1 - G\left(\frac{p_1^2}{2} + W_\delta\right)\right)\psi_t\right\| = \mathcal{O}(t^{-\infty}). \tag{5.48}$$

But since $Gf = 0$, (5.48) implies

$$\begin{aligned}
\left\|g\left(\frac{p_1^2}{2}\right)\psi_t\right\| &\leq \left\|g\left(\frac{p_1^2}{2}\right)G\left(\frac{p_1^2}{2} + W_\delta\right)\psi_t\right\| + \mathcal{O}(t^{-\infty}) \\
&= \left\|g\left(\frac{p_1^2}{2}\right)\left(G\left(\frac{p_1^2}{2} + W_\delta\right) - G(\tilde{H})\right)\psi_t\right\| + \mathcal{O}(t^{-\infty}).
\end{aligned}$$

Let G_1 be an almost analytic extension of $1 - g$. Then

$$\begin{aligned}
&\left(G\left(\frac{p_1^2}{2} + W_\delta\right) - G(\tilde{H})\right)\psi_t \\
&= \frac{1}{\pi}\int \overline{\partial}G_1(z)\left((\tilde{H} - z)^{-1} - \left(\frac{p_1^2}{2} + W_\delta - z\right)^{-1}\right)\psi_t d^2z \\
&= -\pi^{-1}\int \overline{\partial}G_1(z)\left(\frac{p_1^2}{2} + W_\delta - z\right)^{-1}\left[\frac{|p_\perp|^2}{2}(1 - \chi_\delta(p_\perp))\right. \\
&\quad + \left.\tilde{V}_0(x)\left[1 - \chi_\delta\left(\frac{x_\perp}{|x|}\right)\chi(2x)\right]\right](\tilde{H} - z)^{-1}\psi_t d^2z.
\end{aligned} \tag{5.49}$$

Notice that $1 - \chi_\delta(p_\perp) = (1 - \chi_\delta(p_\perp))(1 - \chi_{\frac{1}{2}\delta}(p_\perp))^N$ and that according to Proposition 5.6 $(1 - \chi_{\frac{1}{2}\delta}(p_\perp))\psi_t = \mathcal{O}(t^{-\infty})$. Thus

$$\int \overline{\partial}G_1(z)\left(\frac{p_1^2}{2} + W_\delta - z\right)^{-1}\left[\frac{|p_\perp|^2}{2}(1 - \chi_\delta(p_\perp))(\tilde{H} - z)^{-1}\psi_t\right]d^2z$$

$$= \int \overline{\partial} G_1(z) \left(\frac{p_1^2}{2} + W_\delta - z \right)^{-1} \frac{p_\perp^2}{2} (1 - \chi_\delta(p_\perp))(-1)^N ad^N_{\chi_{\frac{1}{2}\delta}(p_\perp)} (\tilde{H} - z)^{-1} \psi_t d^2 z$$

$$+ \mathcal{O}(t^{-\infty}),$$

and this term is easily seen to be $\mathcal{O}(t^{-N})$. A similar treatment gives the same result for the remaining term in (5.49). Since N is arbitrary,

$$\left\| g\left(\frac{p_1^2}{2}\right) \psi_t \right\| = \mathcal{O}(t^{-\infty}).$$

This gives (5.45) and completes the proof of the proposition. □

Proposition 5.9 *Suppose* $f \in C_0^\infty\left(\left(0, \frac{k_1^2}{2}\right) \setminus (\{2\lambda_1, \ldots, 2\lambda_n\} \cup \sigma_{pp}(\tilde{H}))\right)$, *where* $k_1 = \sqrt{9\lambda_{\min}/2}$ *and* $L > \frac{k_1^2}{2}$. *Then for some* $\delta > 0$,

$$\|z^\alpha \psi_t\| = \mathcal{O}\left(t^{-|\alpha|(\frac{1}{3}+\delta)}\right), \quad |\alpha| \leq 3, \tag{5.50}$$

where $z_j = \left(\frac{x_\perp}{t}\right)_j$, $(p_\perp)_j$, *or* $p_1 - \frac{x_1}{t}$.

Remark. Clearly a more complete result of this type is true, but we will only need (5.50).

Proof. We only sketch the proof because it is similar to that of Proposition 5.6 but much simpler. Let $h \in C_0^\infty((0, k_1) \setminus \{2\sqrt{\lambda_2}, \ldots, 2\sqrt{\lambda_n}\})$ with $h(s) = 1$ if $\frac{s^2}{2} \in \operatorname{supp} f$. Let

$$\begin{aligned}
\gamma(t) &= \left(p_\perp + \beta\left(\frac{x_1}{t}\right)\frac{x_\perp}{t}\right) h\left(\frac{x_1}{t}\right), \\
\tilde{\gamma}(t) &= \left(p_\perp + \tilde{\beta}\left(\frac{x_1}{t}\right)\frac{x_\perp}{t}\right) h\left(\frac{x_1}{t}\right),
\end{aligned}$$

and

$$\Gamma(t) = \sum_{j=2}^n (\gamma_j^* \gamma_j + \tilde{\gamma}_j^* \tilde{\gamma}_j) + h\left(\frac{x_1}{t}\right)\left(p_1 - \frac{x_1}{t}\right)^2 h\left(\frac{x_1}{t}\right).$$

We compute for large t,

$$\begin{aligned}
\mathbf{D}\gamma(t) &= \Big(- \nabla_\perp \tilde{V}_0(x) + \frac{\beta(x_1/t)}{t}\left(p_\perp - \frac{x_\perp}{t}\right) \\
&\quad + (2t)^{-1}\left(\beta'\left(\frac{x_1}{t}\right)\left(p_1 - \frac{x_1}{t}\right) + \left(p_1 - \frac{x_1}{t}\right)\beta'\left(\frac{x_1}{t}\right)\right)\left(\frac{x_\perp}{t}\right)\Big) h\left(\frac{x_1}{t}\right) \\
&\quad + \left(p_\perp + \beta\left(\frac{x_1}{t}\right)\left(\frac{x_\perp}{t}\right)\right)\mathbf{D}h\left(\frac{x_1}{t}\right) \\
&= t^{-1}\beta\left(\frac{x_1}{t}\right)\gamma(t) + \mathcal{E}_\gamma;
\end{aligned} \tag{5.51}$$

$$
\begin{aligned}
\mathcal{E}_\gamma &= t^{-1}\left(\frac{t}{x_1}\right)^3 h\left(\frac{x_1}{t}\right)\int_0^1 (1-\theta)\tilde{V}_0^{(3)}(1,\theta u)\left(\frac{x_\perp}{t}\right)^{(2)} d\theta \\
&\quad + (2t)^{-1}\left(\beta'\left(\frac{x_1}{t}\right)\left(p_1-\frac{x_1}{t}\right)+\left(p_1-\frac{x_1}{t}\right)\beta'\left(\frac{x_1}{t}\right)\right)\left(\frac{x_\perp}{t}\right)h\left(\frac{x_1}{t}\right) \\
&\quad + \left(p_\perp+\beta\left(\frac{x_1}{t}\right)\left(\frac{x_\perp}{t}\right)\right)\mathbf{D}h\left(\frac{x_1}{t}\right),
\end{aligned} \tag{5.52}
$$

where $u = x_\perp/x_1$ and

$$
\begin{aligned}
(\tilde{V}_0^{(3)}(1,a)b^{(2)})_j &= \sum_{k,\ell\geq 2}(\partial_j\partial_k\partial_\ell\tilde{V}_0)(1,a)b_kb_\ell; \quad j\geq 2 \\
&= 0, \qquad j=1.
\end{aligned}
$$

Also,

$$
\mathbf{D}\left(p_1-\frac{x_1}{t}\right) = -t^{-1}\left(p_1-\frac{x_1}{t}\right)-\partial_1\tilde{V}_0(x), \tag{5.53}
$$

and for $x_1 > \frac{1}{2}$,

$$
-\partial_1\tilde{V}_0(x) = t^{-1}\left(\frac{t}{x_1}\right)^3\int_0^1\left\langle\frac{x_\perp}{t},\tilde{V}_0^{(2)}(1,\theta u)\frac{x_\perp}{t}\right\rangle d\theta. \tag{5.54}
$$

We can also compute

$$
\begin{aligned}
\frac{x_\perp}{t} &= \left(1-h\left(\frac{x_1}{t}\right)\right)\left(\frac{x_\perp}{t}\right)+\left(\beta\left(\frac{x_1}{t}\right)-\tilde{\beta}\left(\frac{x_1}{t}\right)\right)^{-1}(\gamma(t)-\tilde{\gamma}(t)) \\
p_\perp &= \left(1-h\left(\frac{x_1}{t}\right)\right)p_\perp+\left(\beta\left(\frac{x_1}{t}\right)-\tilde{\beta}\left(\frac{x_1}{t}\right)\right)^{-1}\left(\beta\left(\frac{x_1}{t}\right)\tilde{\gamma}(t)-\tilde{\beta}\left(\frac{x_1}{t}\right)\gamma(t)\right),
\end{aligned} \tag{5.55}
$$

where it must be remembered that both $\gamma(t)$ and $\tilde{\gamma}(t)$ contain a factor of $h\left(\frac{x_1}{t}\right)$ which is zero if $\left(\beta\left(\frac{x_1}{t}\right)-\tilde{\beta}\left(\frac{x_1}{t}\right)\right)$ is not invertible. Thus (5.55) makes sense if properly interpreted.

We calculate

$$
\begin{aligned}
\frac{d}{dt}(\psi_t,\Gamma(t)\psi_t) &= t^{-1}\sum_{j=2}^n\left(\psi_t,\left[\gamma_j^*2\ \mathrm{Re}\ \beta_j\left(\frac{x_1}{t}\right)\gamma_j+\tilde{\gamma}_j^*2\ \mathrm{Re}\ \hat{\beta}_j\left(\frac{x_1}{t}\right)\tilde{\gamma}_j\right]\psi_t\right) \\
&\quad -\frac{2}{t}\left(\psi_t,h\left(\frac{x_1}{t}\right)\left(p_1-\frac{x_1}{t}\right)^2h\left(\frac{x_1}{t}\right)\psi_t\right) \\
&\quad + \text{Error terms} \\
&\leq -\left(\frac{2}{3}+3\delta\right)t^{-1}(\psi_t,\Gamma(t)\psi_t)+\text{Error terms}.
\end{aligned}
$$

Here we use $2\ \mathrm{Re}\ \beta_j\left(\frac{x_1}{t}\right) \leq -\left(\frac{2}{3}+3\delta\right)$ in the support of $h\left(\frac{x_1}{t}\right)$ for some small $\delta > 0$. We choose δ small enough so that $\frac{2}{3}+3\delta < 2$. We have used (5.51) and a similar formula for $\mathbf{D}\tilde{\gamma(t)}$ as well as (5.53). All terms which involve $\mathcal{E}_\gamma$, $\mathcal{E}_{\tilde{\gamma}}$, and $-\partial_1 \tilde{V}_0(x)$ have been put into the "Error terms". In treating these terms we note

(i) all terms involving $\mathbf{D}h\left(\frac{x_1}{t}\right)$, $h'\left(\frac{x_1}{t}\right)$, or $\left(1-h\left(\frac{x_1}{t}\right)\right)$ contribute $\mathcal{O}(t^{-\infty})$ because of Proposition 5.8;

(ii) from (5.52) and (5.54) we see that the remaining error terms are cubic in the components of z and contain a factor of t^{-1}. Thus using (i), (5.55), and Proposition 5.6, these terms can be estimated up to an error $\mathcal{O}(t^{-\infty})$ by $\frac{\delta}{t}(\psi_t, \Gamma(t)\psi_t)$.

This gives

$$\begin{aligned}
\frac{d}{dt}(\psi_t, \Gamma(t)\psi_t) &\leq -\left(\frac{2}{3}+2\delta\right) t^{-1}(\psi_t, \Gamma(t)\psi_t) + \mathcal{O}(t^{-\infty}), \\
\text{and} \quad (\psi_t, \Gamma(t)\psi_t) &= \mathcal{O}\left(t^{-\left(\frac{2}{3}+2\delta\right)}\right). \qquad (5.56)
\end{aligned}$$

Using (5.55) again we obtain (5.50) for $|\alpha| = 1$.

Next consider

$$\begin{aligned}
\frac{d}{dt}(\psi_t, \Gamma(t)^2\psi_t) &= 2\ \mathrm{Re}\ (\psi_t, \Gamma(t)\mathbf{D}\Gamma(t)\psi_t) \\
&= 2\ \mathrm{Re}\Big\{\Big(\psi_t, \Big[\sum_{j=2}^{n}\Big(\gamma_j^*(t)\mathbf{D}\Gamma(t)\gamma_j + \tilde{\gamma}_j^*(t)\mathbf{D}\Gamma(t)\tilde{\gamma}_j(t)\Big) \\
&\quad + h\left(\frac{x_1}{t}\right)\left(p_1 - \frac{x_1}{t}\right)\mathbf{D}\Gamma(t)\left(p_1 - \frac{x_1}{t}\right)h\left(\frac{x_1}{t}\right) \\
&\quad + \left[\mathbf{D}\Gamma(t), h\left(\frac{x_1}{t}\right)\left(p_1 - \frac{x_1}{t}\right)\right] \\
&\quad + \sum_{j=2}^{n}\Big(\Big[\mathbf{D}\Gamma(t), \gamma_j^*(t)\Big]\gamma_j(t) + \Big[\mathbf{D}\Gamma(t), \tilde{\gamma}_j^*(t)\Big]\gamma_j(t)\Big)\Big]\psi_t\Big)\Big\}.
\end{aligned}$$

From (5.50) for $|\alpha| = 1$, (5.51), and (5.52) we see that the commutator terms contribute $\mathcal{O}\left(t^{-2}t^{-\left(\frac{2}{3}+2\delta\right)}\right)$. We now estimate $\mathbf{D}\Gamma(t)$ as above, and then restore the original order of the operators incurring another $\mathcal{O}\left(t^{-2}t^{-\left(\frac{2}{3}+2\delta\right)}\right)$ error. We obtain

$$\begin{aligned}
\frac{d}{dt}(\psi_t, \Gamma(t)^2\psi_t) &\leq -2\left(\frac{2}{3}+2\delta\right) t^{-1}(\psi_t, (\Gamma(t))^2\psi_t) \\
&\quad + \mathcal{O}\left(t^{-2}t^{-\left(\frac{2}{3}+2\delta\right)}\right).
\end{aligned}$$

Upon integration this gives

$$(\psi_t, \Gamma(t)^2\psi_t) = \mathcal{O}\left(t^{-4(\frac{1}{3}+\delta)}\right) \tag{5.57}$$

if we demand $\frac{2}{3} + 2\delta < 1$.

The operator $\Gamma(t)$ is essentially a sum of squares of self-adjoint operators. A short calculation gives

$$\begin{aligned}\Gamma(t) &= 2h\left(\frac{x_1}{t}\right)^2\left(p_\perp - \frac{x_\perp}{2t}\right)^2 + 2^{-1}\left\langle \frac{x_\perp}{t}, h\left(\frac{x_1}{t}\right)^2 \left| I - 4\left(\frac{t}{x_1}\right)^2 \lambda \right| \frac{x_\perp}{t}\right\rangle \\ &\quad + h\left(\frac{x_1}{t}\right)\left(p_1 - \frac{x_1}{t}\right)^2 h\left(\frac{x_1}{t}\right).\end{aligned} \tag{5.58}$$

The last term is not the square of a simple self-adjoint operator but because of Proposition 5.8, this causes no problems. We now use the method of proof of Proposition 5.6 (see Eqn. (5.29)) to show that (5.57) implies (5.50) for $|\alpha| = 2$. This again requires $\frac{1}{3} + \delta < \frac{1}{2}$. Finally we calculate

$$\frac{d}{dt}(\psi_t, \Gamma(t)^3\psi_t) = 3\left(\psi_t, \Gamma(t)\mathbf{D}\Gamma(t)\Gamma(t)\psi_t\right) + \left(\psi_t, \left[\Gamma(t), \left[\Gamma(t), \mathbf{D}\Gamma(t)\right]\right]\psi_t\right).$$

The commutator term is seen to contribute $\mathcal{O}\left(t^{-3}t^{-(\frac{2}{3}+2\delta)}\right)$ which again leads to

$$(\psi_t, \Gamma(t)^3\psi(t)) = \mathcal{O}\left(t^{-6(\frac{1}{3}+\delta)}\right).$$

Using the sum of squares argument from the proof of Proposition 5.6 once more, we obtain (5.50) for $|\alpha| = 3$. □

Proposition 5.10 *Suppose $f \in C_0^\infty\left(\left(\frac{k_2^2}{2}, L\right) \setminus E\right)$ where $E = \sigma_{pp}(\tilde{H}) \cup \left\{\frac{s^2}{2} : s \in \mathcal{R}\right\}$, and $k_2 = 2\sqrt{\lambda_{\max}}$. Let $h_1 \in C_0^\infty((k_2, \infty)\setminus\mathcal{R})$ with $h_1(s) = 1$ if $\frac{s^2}{2} \in \operatorname{supp} f$ and let $h_2 \in C_0^\infty(B_\delta)$, $B_\delta = \{y \in \mathbb{R}^{n-1} : |y| < \delta\}$, with $h_2 = 1$ on $\frac{B_\delta}{2}$. Set $h(t,x) = h_1\left(\frac{x_1}{t}\right)h_2\left(\frac{x_\perp}{\delta t}\right)$. Then for small enough $\delta > 0$, there is a $\theta > 0$ so that*

$$\left\| |p - h(t,x)\nabla_x S(t,x)|^j \psi_t \right\| = \mathcal{O}\left(t^{-(1+\theta)j/2}\right), \quad j = 1, 2. \tag{5.59}$$

Proof. As an easy consequence of Propositions 5.6 and 5.8,

$$\|(1-h)\psi_t\| + \|(\partial_{t,x}^\alpha h)\psi_t\| = \mathcal{O}(t^{-\infty})$$

for any multi-index α with $|\alpha| \geq 1$. We compute for small δ,

$$\mathbf{D}(p - h\nabla_x S) = -\mathbf{D}h\nabla_x S - \nabla\tilde{V}_0(x) - h\left\{\nabla_x\partial_t S + \frac{S^{(2)}p + pS^{(2)}}{2}\right\},$$

where $S^{(2)}(t,x)_{ij} = \partial_{x_i}\partial_{x_j} S(t,x)$. Using the Hamilton-Jacobi equation,

$$-\partial_t S = \frac{(\nabla_x S)^2}{2} + V_0(x),$$

which holds in supp h if δ is small enough, we obtain (assuming δ is small enough and t large enough so that $h(\nabla\tilde{V}_0 - \nabla V_0) = 0$)

$$\mathbf{D}(p - h\nabla_x S) = -\mathbf{D}h\nabla_x S + (h-1)\nabla\tilde{V}_0 - h(S^{(2)}(p - \nabla_x S) + (p - \nabla_x S)S^{(2)}/2.$$

We thus compute

$$\begin{aligned}
&\frac{d}{dt}(\psi_t, (p - h\nabla_x S)^2\psi_t) \\
&= \quad \mathrm{Re}(\psi_t, -h\langle S^{(2)}(p - h\nabla_x S) + (p - h\nabla_x S)S^{(2)}, p - h\nabla_x S\rangle\psi_t) + \mathcal{O}(t^{-\infty}) \\
&= \quad -2\left(\psi_t \left[\langle p - h\nabla_x S, hS^{(2)}(p - h\nabla_x S)\rangle - \frac{h}{4}\Delta_x^2 S\right]\psi_t\right) + \mathcal{O}(t^{-\infty}).
\end{aligned}$$

Using (2.12) and the fact that $S(t,x) = tS\left(1, \frac{x}{t}\right)$ has Taylor series in $\frac{x_\perp}{t}$ with coefficients depending on $\frac{x_1}{t}$ we obtain

$$S(t,x) = t\left(\frac{1}{2}\left(\frac{x_1}{t}\right)^2 - \frac{1}{2}\left\langle\frac{x_\perp}{t}, \tilde{\beta}\left(\frac{x_1}{t}\right)\frac{x_\perp}{t}\right\rangle + \mathcal{O}\left(\left(\frac{|x_\perp|}{t}\right)^3\right)\right) \tag{5.60}$$

near $\frac{x_\perp}{t} = 0$. Thus

$$S^{(2)}(t, x_1, 0) = t^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -\tilde{\beta}\left(\frac{x_1}{t}\right)\end{pmatrix},$$

and by the homogeneity property of S,

$$h\Delta_x^2 S = \mathcal{O}(t^{-3}).$$

We recall that

$$\tilde{\beta}(k) = \frac{-1 - \sqrt{1 - 4\lambda/k^2}}{2},$$

and thus since h_1 has compact support in (k_2, ∞), there is a $\theta > 0$ so that for δ small enough

$$S^{(2)}(t,x) \geq (2t)^{-1}(1+\theta)I$$

in supp h. It follows that

$$\frac{d}{dt}(\psi_t, (p - h\nabla_x S)^2\psi_t) \leq -\left(\frac{1+\theta}{t}\right)(\psi_t, (p - h\nabla_x S)^2\psi_t) + \mathcal{O}(t^{-3}),$$

and thus (5.59) follows for $j = 1$.

Now we consider

$$
\begin{aligned}
&\frac{d}{dt}(\psi_t, [(p-h\nabla_x S)^2]^2\psi_t) \\
&= \; 2 \text{ Re} \left(\psi_t, \left[\mathbf{D}(p-h\nabla_x S)^2\right](p-h\nabla_x S)^2\psi_t\right) \\
&= \; -4 \text{ Re} \left(\psi_t, \left[\langle p-h\nabla_x S, hS^{(2)}(p-h\nabla_x S)\rangle - \frac{h}{4}\Delta_x^2 S\right](p-h\nabla_x S)^2\psi_t\right) + \mathcal{O}(t^{-\infty}).
\end{aligned}
$$

Let $\gamma_j(t) = (p-h\nabla_x S)_j$. Then using (for self-adjoint L_0),

$$\text{Re } L_0(p-h\nabla_x S)^2 = \sum_j \gamma_j L_0 \gamma_j + \frac{1}{2}\sum_j [\gamma_j, [\gamma_j, L_0]],$$

with $L_0 = \langle p-h\nabla_x S, hS^{(2)}(p-h\nabla_x S)\rangle - \frac{h}{4}\Delta_x^2 S$ and the fact that γ_j and γ_k commute up to terms involving derivatives of h, we obtain

$$
\begin{aligned}
\frac{d}{dt}(\psi_t, (\gamma^2)^2\psi_t) &= -4\sum_j \left(\gamma_j\psi_t, \left(\langle\gamma, hS^{(2)}\gamma\rangle - \frac{h}{4}\Delta_x^2 S\right)\gamma_j\psi_t\right) \\
&\quad + \left(\psi_t, \left[2\langle\gamma, h\Delta_x S^{(2)}\gamma\rangle + \frac{h}{2}\Delta_x^3 S\right]\psi_t\right) + \mathcal{O}(t^{-\infty}) \\
&\leq -\frac{2(1+\theta)}{t}\sum_j (\psi_t, \gamma_j\gamma^2\gamma_j\psi_t) + \mathcal{O}(t^{-1-\theta}t^{-3}) \\
&= -\frac{2(1+\theta)}{t}(\psi_t, (\gamma^2)^2\psi_t) + \mathcal{O}(t^{-4-\theta}),
\end{aligned}
$$

where we have used $|\Delta_x^2 S| + |\Delta_x S^{(2)}| = \mathcal{O}(t^{-3})$ and $|\Delta_x^3 S| = \mathcal{O}(t^{-5})$ in supp h and (5.59) for $j=1$. Integration gives (5.59) for $j=2$. $\square$

6 Completeness of the wave operators

In this section we prove the completeness parts of Theorem 3.1 and 3.2. The existence parts of these theorems have already been given by combining Theorems 4.1 and 4.2 with Lemma 3.6. Theorems 4.1 and 4.2 also prove the intertwining property of the wave operators.

Completion of the proof of Theorem 3.1. We first introduce a more convenient notation. We choose some $L > \frac{k_1^2}{2}$ and denote by V_L the potential constructed in Section 3, which we have been calling $\tilde{V}_0$. We write $H_L = \frac{1}{2}p^2 + V_L$ and denote the wave operator, $\tilde{W}$, of Theorem 3.7 by W_L.

It follows easily from the proof of Theorem 4.1 that

$$\Omega_L = \lim_{t\to\infty} e^{itH_L} U_0(t)$$

on $\mathcal{H}_1 = \chi_{[0,k_1]}(p_1)L^2(\mathbb{R}^n)$ and satisfies the intertwining relation

$$e^{itH_L}\Omega_L = \Omega_L e^{\frac{itp_1^2}{2}}. \tag{6.1}$$

The proof of Theorem 4.1 needs only to be supplemented by the remark that for f as in the proof of that theorem

$$\|(V_L(x) - V_0(x))U_0(t)f\| = \mathcal{O}(t^{-\infty}).$$

This follows from Lemmas 4.2 and 4.3. It also follows from Theorem 3.7 that

$$W_L\Omega_L = \Omega, \tag{6.2}$$

and that

$$W_L : \text{Ran } P_{e_1}^{H_L} \cap E_{H_L}\left(\left(0, \frac{k_1^2}{2}\right)\right) \overset{\text{onto}}{\longrightarrow}$$

$$\text{Ran } P_{e_1}^{H} \cap E_{H}\left(\left(0, \frac{k_1^2}{2}\right)\right) = \mathcal{H}_2.$$

We thus need to show that

$$\Omega_L : \mathcal{H}_1 \overset{\text{onto}}{\longrightarrow} \mathcal{H}_{2,L} \equiv \text{Ran } P_{e_1}^{H_L} \cap E_{H_L}\left(\left(0, \frac{k_1^2}{2}\right)\right). \tag{6.3}$$

From (6.1) and the definition of Ω_L, it easily follows that

$$\text{Ran } \Omega_L \subset \mathcal{H}_{2,L}.$$

We will show that considered as a map from $\mathcal{H}_1$ into $\mathcal{H}_{2,L}$, ker $\Omega_L^* = \{0\}$ so that $\overline{\text{Ran } \Omega_L} = \text{Ran } \Omega_L = \mathcal{H}_{2,L}$. Suppose $\psi = f(H_L)\phi$ where

$$f \in C_0^\infty\left(\left(0, \frac{k_1^2}{2}\right) \setminus \{2\sqrt{\lambda_2}, \ldots, 2\sqrt{\lambda_n}\} \cup \sigma_{pp}(H_L)\right)$$

and $\phi \in \mathcal{S}(\mathbb{R}^n)$. If $g \in \mathcal{H}_1$,

$$\begin{aligned}(g, \Omega_L^*\psi) &= \lim_{t\to\infty}(U_0(t)g, e^{-itH_L}\psi) \\ &= \lim_{t\to\infty}(g, U_0(t)^{-1}e^{-itH_L}\psi). \end{aligned} \tag{6.4}$$

Using Lemma 4.3 in conjunction with Propositions 5.1, 5.2, we know that if $0 \le \chi \le 1$, $\chi \in C^\infty(\mathbb{R})$, supp $\chi \subset (1, \infty)$ and $\chi(s) = 1$ if $s \ge 2$, then for small enough $\delta > 0$,

$$\begin{aligned}\left(1 - \chi\left(\frac{x_1}{\delta t}\right)\right)\psi_t &= \mathcal{O}(t^{-\infty}); \\ \left(1 - \chi\left(\frac{p_1}{\delta}\right)\right)\psi_t &= \mathcal{O}(t^{-\infty}),\end{aligned}$$

and similarly for any derivative of χ. Here $\psi_t = e^{-itH_L} f(H_L)\phi$. We thus have

$$U_0(t)^{-1}\psi_t = U_0(t)^{-1}\chi\Big(\frac{p_1}{\delta}\Big)\psi_t + \mathcal{O}(t^{-\infty}).$$

We use Cook's method to prove the convergence of $U_0(t)^{-1}\Big(\frac{p_1}{\delta}\Big)\psi_t$:

$$\begin{aligned}
&\int_T^\infty \left\| \frac{d}{dt}\left(U_0(t)^{-1}\chi\left(\frac{p_1}{\delta}\right)\psi_t\right)\right\| dt \\
&\qquad = \int_T^\infty \left\| \left(\frac{\langle x_\perp, \lambda x_\perp\rangle}{2(tp_1)^2}\chi\Big(\frac{p_1}{\delta}\Big) - \chi\Big(\frac{p_1}{\delta}\Big)V_L(x)\right)\psi_t\right\| dt. \qquad (6.5)
\end{aligned}$$

As in the proof of Theorem 4.1 we write

$$\begin{aligned}
\frac{\langle x_\perp, \lambda x_\perp\rangle}{2(tp_1)^2}\chi\Big(\frac{p_1}{\delta}\Big)\psi_t &= \chi\Big(\frac{x_1}{\delta t}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{2(tp_1)^2}\chi\Big(\frac{p_1}{\delta}\Big)\psi_t + \mathcal{O}(t^{-\infty}) \\
&= \chi\Big(\frac{x_1}{\delta t}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{x_1^2}\chi\Big(\frac{p_1}{\delta}\Big)\psi_t \\
&\quad + \chi\Big(\frac{x_1}{\delta t}\Big)\Big(\frac{t}{x_1}\Big)^2\left(\Big(\frac{x_1}{t}\Big)^2 - p_1^2\right)\frac{1}{p_1^2}\chi\Big(\frac{p_1}{\delta}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}\psi_t \\
&\quad + \mathcal{O}(t^{-\infty}).
\end{aligned}$$

We have

$$\begin{aligned}
&\chi\Big(\frac{x_1}{\delta t}\Big)\Big(\frac{t}{x_1}\Big)^2\left(p_1^2 - \Big(\frac{x_1}{t}\Big)^2\right)\frac{1}{p_1^2}\chi\Big(\frac{p_1}{\delta}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2} \\
&\quad = \chi\Big(\frac{x_1}{\delta t}\Big)\Big(\frac{t}{x_1}\Big)^2 p_1\left(p_1 - \frac{x_1}{t}\right)\frac{1}{p_1^2}\chi\Big(\frac{p_1}{\delta}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2} \\
&\quad + \chi\Big(\frac{x_1}{\delta t}\Big)\Big(\frac{t}{x_1}\Big)\left(p_1 - \frac{x_1}{t}\right)\frac{1}{p_1^2}\chi\Big(\frac{p_1}{\delta}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2} \\
&\quad - i\chi\Big(\frac{x_1}{\delta t}\Big)\Big(\frac{t}{x_1}\Big)^2\frac{1}{p_1^2}\chi\Big(\frac{p_1}{\delta}\Big)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^3}, \qquad (6.6)
\end{aligned}$$

where we have used $p_1^2 - \left(\frac{x_1}{t}\right)^2 = p_1\left(p_1 - \frac{x_1}{t}\right) + \frac{x_1}{t}\left(p - \frac{x_1}{t}\right) - it^{-1}$. But

$$\begin{aligned}
&\left\|\left(p_1 - \frac{x_1}{t}\right)\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^2}\psi_t\right\| + \left\|\frac{\langle x_\perp, \lambda x_\perp\rangle}{t^3}\psi_t\right\| \\
&\quad = \mathcal{O}(t^{-1-3\delta}) + \mathcal{O}(t^{-\frac{5}{3}-2\delta}).
\end{aligned}$$

So the contribution of (6.6) to (6.5) is finite. Thus we must show

$$\left\| \left(\chi\Big(\frac{x_1}{\delta t}\Big) \frac{\langle x_\perp, \lambda x_\perp \rangle}{x_1^2} \chi\Big(\frac{p_1}{\delta}\Big) - \chi\Big(\frac{p_1}{\delta}\Big) V_L(x) \right) \psi_t \right\| \tag{6.7}$$

is integrable on $[T, \infty)$. We write

$$\chi\Big(\frac{p_1}{\delta}\Big) V_L(x) \psi_t = \chi\Big(\frac{p_1}{\delta}\Big) V_L(x) \chi\Big(\frac{x_1}{\delta t}\Big) \psi_t + \mathcal{O}(t^{-\infty}),$$

and use the commutator expansion (5.25) to obtain

$$\chi\Big(\frac{p_1}{\delta}\Big) V_L(x) \psi_t = V_L(x) \chi\Big(\frac{x_1}{\delta t}\Big) \chi\Big(\frac{p_1}{\delta}\Big) \psi_t + \mathcal{O}(t^{-\infty}).$$

Thus up to an error of $\mathcal{O}(t^{-\infty})$, (6.7) can be written

$$\begin{aligned} & \left\| \chi\Big(\frac{x_1}{\delta t}\Big) \Big(\frac{\langle x_\perp, \lambda x_\perp \rangle}{x_1^2} - V_L(x) \Big) \chi\Big(\frac{p_1}{\delta}\Big) \psi_t \right\| \\ = & \left\| \chi\Big(\frac{x_1}{\delta t}\Big) \Big(\frac{\langle x_\perp, \lambda x_\perp \rangle}{x_1^2} - V_L\Big(1, \frac{x_\perp}{x_1}\Big) \Big) \psi_t \right\| + \mathcal{O}(t^{-\infty}). \end{aligned} \tag{6.8}$$

Clearly,

$$\left| \chi\Big(\frac{x_1}{\delta t}\Big) \Big(\frac{\langle x_\perp, \lambda x_\perp \rangle}{x_1^2} - V_L\Big(1, \frac{x_\perp}{x_1}\Big) \Big) \right| \leq C \left| \frac{x_\perp}{t} \right|^3,$$

and thus from Proposition 5.9, (6.8) is integrable. It follows that for some $g_1 \in L^2(\mathbb{R}^n)$,

$$s - \lim_{t \to \infty} U_0(t)^{-1} e^{-itH_L} \psi = g_1. \tag{6.9}$$

We now need to show that $g_1 \in \mathcal{H}_1$, and for this we need some version of the intertwining property for the inverse wave operator.

We first show that

$$s - \lim_{t \to \infty} U_0(t)^{-1} U_0(t+s) = e^{\frac{-isp_1^2}{2}}. \tag{6.10}$$

As in the proof of Theorem 4.1 we choose f_1 from the dense subset of $L^2(\mathbb{R}^n)$,

$$\{f_1 : \hat{f}_1 \in C_0^\infty((0, \infty) \setminus \mathcal{E} \times \mathbb{R}^{n-1})\}$$

and write $\overline{U}_0(t) = e^{ip_1^2(t-1)/2} U_0(t)$. Then

$$U_0(t)^{-1} U_0(t+s) = e^{-i(p_1^2/2)s} \overline{U}_0(t)^{-1} \overline{U}_0(t+s).$$

We obtain

$$\overline{U}_0(t)^{-1} \overline{U}_0(t+s) f_1 = f_1 - i \int_0^s \overline{U}_0(t)^{-1} \left(\frac{p_\perp^2}{2} + \frac{\langle x_\perp, \lambda x_\perp \rangle}{2p_1^2 (t+\theta)^2} \right) \overline{U}_0(t+\theta) f_1 d\theta, \tag{6.11}$$

and from (4.1) and (4.2), the second term on the right side of (6.11) has limit zero as $t \to \infty$. This proves (6.10).

From (6.9) and (6.10) we obtain

$$s - \lim_{t\to\infty} U_0(t)^{-1} e^{-itH_L} e^{-isH_L} \psi = e^{-isp_1^2/2} g_1,$$

and a simple approximation argument gives

$$s - \lim_{t\to\infty} U_0(t)^{-1} e^{-itH_L} h(H_L)\psi = h\Big(\frac{p_1^2}{2}\Big) g_1,$$

for example, for $h \in C_0^\infty\Big(\Big(\frac{k_1^2}{2}, \infty\Big)\Big)$. This shows $\chi_{[k_1,\infty)}(|p_1|)g_1 = 0$.

But we also know that

$$\psi_t = \chi\Big(\frac{p_1}{\delta}\Big)\psi_t + \mathcal{O}(t^{-\infty}),$$

and $U_0(t)^{-1}$ commutes with $\chi\Big(\frac{p_1}{\delta}\Big)$ so that

$$\chi\Big(\frac{p_1}{\delta}\Big) g_1 = g_1.$$

Thus $g_1 \in \mathcal{H}_1$, and it follows that

$$\Omega_L^* \psi = g_1.$$

But from (6.9), $\|g_1\| = \|\psi\|$, and since ψ was chosen from a dense subset of $\mathcal{H}_{2,L}$, we see that Ω_L^* is isometric so that $\ker \Omega_L^* = \{0\}$.

This completes the proof. □

Completion of the proof of Theorem 3.2. The proof of the existence of $\tilde{\Omega}$ (Theorem 4.2) also yields the existence of $\tilde{\Omega}_L : \tilde{\mathcal{H}}_1 = L^2((k_2,\infty) \times \mathbb{R}^{n-1}) \to L^2(\mathbb{R}^n)$, first defined on $f \in C_0^\infty((k_2,\infty) \setminus \mathcal{R} \times \mathbb{R}^{n-1})$ as

$$\lim_{t\to\infty} e^{itH_L} \tilde{U}_0(t) f,$$

and then extended by continuity to an isometric operator on $\tilde{\mathcal{H}}_1$. The proof is exactly the same because the term involving $V_0(x) - V_L(x)$ is actually zero for large t. We also have the intertwining relation

$$e^{itH_L}\tilde{\Omega}_L = \tilde{\Omega}_L e^{itk^2/2}. \tag{6.12}$$

The support properties of $\tilde{U}_0(t)f$ show that $\left\|\frac{x_\perp}{x_1}\tilde{U}_0(t)f\right\| = \mathcal{O}(t^{-\delta})$ for some $\delta > 0$ which shows

$$\begin{aligned} \text{Ran } \tilde{\Omega} &\subset \text{Ran } P_{e_1}^{H}; \\ \text{Ran } \tilde{\Omega}_L &\subset \text{Ran } P_{e_1}^{H_L}. \end{aligned} \tag{6.13}$$

Combined with (6.12) we obtain

$$\tilde{\Omega}_L : L^2((k_2, \sqrt{2L}) \times \mathbb{R}^{n-1}) \longrightarrow P_{e_1}^{H_L} E_{H_L}\left(\left(\frac{k_2^2}{2}, L\right)\right) L^2(\mathbb{R}^n). \tag{6.14}$$

From Lemma 3.7 and the obvious intertwining property of $\tilde{W}_L = s - \lim e^{itH} e^{-itH_L}$ defined on Ran $P_{e_1}^{H_L}$, we find that $\tilde{W}_L$ is a unitary operator

$$\begin{aligned} \tilde{W}_L : P_{e_1}^{H_L} E_{H_L}\left(\left(\tfrac{k_2^2}{2}, L\right)\right) L^2(\mathbb{R}^n) &\stackrel{\text{onto}}{\longrightarrow} \\ P_{e_1}^{H} E_{H}\left(\left(\tfrac{k_2^2}{2}, L\right)\right) L^2(\mathbb{R}^n). \end{aligned} \tag{6.15}$$

In addition, it easily follows that

$$\tilde{\Omega} = W_L \tilde{\Omega}_L.$$

Thus to show Ran $\tilde{\Omega} \supset P_{e_1}^{H} E_{H}\left(\left(\frac{k_2^2}{2}, L\right)\right) L^2(\mathbb{R}^n)$, it is enough to show that $\tilde{\Omega}_L$ considered as a map as in (6.14) has ker $\Omega_L^* = \{0\}$. The set

$$\begin{aligned} &\left\{ f(H_L)\phi : \ \phi \in \mathcal{S}(\mathbb{R}^n), f \in C_0^\infty\left(\left(\tfrac{k_2^2}{2}, L\right) \setminus E\right)\right\}, \\ &E = \sigma_{pp}(H_L) \cup \left\{\tfrac{s^2}{2} : \ s \in \mathcal{R}\right\} \end{aligned} \tag{6.16}$$

is dense in $P_{e_1}^{H_L} E_{H_L}\left(\left(\frac{k_2^2}{2}, L\right)\right) L^2(\mathbb{R}^n)$ since $V_L(-e_1) > L$ implies

$$P_{e_1}^{H_L} E_{H_L}\left(\left(\frac{k_2^2}{2}, L\right)\right) = P_{\mathrm{cont}}(H_L) E_{H_L}\left(\left(\frac{k_2^2}{2}, L\right)\right).$$

Thus choose $\psi = f(H_L)\phi$ as in (6.16) and h_1, h_2, h, and δ as in Proposition 5.10. Let $g \in C_0^\infty((k_2, \sqrt{2L}) \setminus \mathcal{R} \times \mathbb{R}^{n-1})$. Then

$$\begin{aligned} (\tilde{\Omega}_L g, \psi) = (g, \tilde{\Omega}_L^* \psi) &= \lim_{t\to\infty} (\tilde{U}_0(t) g, e^{-itH_L}\psi) \\ &= \lim_{t\to\infty} (\tilde{U}_0(t) g, h\psi_t), \end{aligned} \tag{6.17}$$

where $\psi_t = e^{-itH_L}\psi$. The insertion of h in (6.12) is justified because from Propositions 5.6 and 5.8, $(1-h)\psi_t = \mathcal{O}(t^{-\infty})$. If δ is chosen small enough, according to Lemma 2.5, $h\psi_t$ is in the domain of $\tilde{U}_0(t)^{-1}$, and we can differentiate $\tilde{U}_0(t)^{-1} h\psi_t$:

$$\begin{aligned} i\partial_t(\tilde{U}_0(t)^{-1} h\psi_t) &= \tilde{U}_0(t)^{-1}\left\{\frac{1}{2}(p - \nabla_x S(t,s))^2 h - \left[\frac{p^2}{2}, h\right] - \mathbf{D}h\right\}\psi_t \\ &= \frac{1}{2}\tilde{U}_0(t)^{-1} h(p - h\nabla_x S(t,x))^2 \psi_t + \mathcal{O}(t^{-\infty}). \end{aligned} \tag{6.18}$$

According to Proposition 5.10, (6.18) has an integrable norm so that Cook's method gives the existence of the strong limit

$$s-\lim_{t\to\infty}\tilde{U}_0(t)^{-1}h\psi_t=g_1.$$

From (6.17) we have

$$(g,\tilde{\Omega}_L^*\psi)=(g,g_1). \tag{6.19}$$

We need to show $g_1\in L^2((k_2,\sqrt{2L})\times\mathbb{R}^{n-1})$. Note first that $g_1\in L^2((0,\infty)\times\mathbb{R}^{n-1})$ since $U_0(t)^{-1}h\psi_t$ is in the domain of $\tilde{U}_0(t)$ which is contained in $L^2((0,\infty)\times\mathbb{R}^{n-1})$. Consider the limit

$$\lim_{t\to\infty}\tilde{U}_0(t)^{-1}he^{-isH_L}\psi_t. \tag{6.20}$$

For large t and small δ,

$$\begin{aligned}
&h\left(e^{-isH_L}-e^{-is(|\nabla_x S(t,x)|^2/2+V_L(x))}\right)\psi_t\\
&\quad=\left(\frac{h}{2i}\right)\int_0^s e^{-i(s-\tau)(|\nabla_x(S(t,x)|^2/2+V_L(x))}(p^2-|\nabla_x S(t,x)|^2)e^{-i\tau H_L}\psi_t d\tau,
\end{aligned}$$

where we have used $V_0(x)=V_L(x)$ in supp h if t is large and δ small. According to Propositions 5.10 and 5.8,

$$\|h(p^2-|\nabla_x S(t,x)|^2)e^{-i\tau H_L}\psi_t\|=\mathcal{O}(t^{-(1+\theta)/2}),$$

for each τ (note $f(\xi)e^{-i\tau\xi}$ is another C_0^∞ function with the right support properties) and thus by the dominated convergence theorem

$$\lim_{t\to\infty}\int_0^s\|h(p^2-|\nabla_x S(t,x)|^2)e^{-i\tau H_L}\psi_t\|d\tau=0.$$

By the definition of $k\left(\frac{x}{t}\right)$, it follows that

$$\lim_{t\to\infty}\tilde{U}_0(t)^{-1}he^{-isH_L}\psi_t=e^{-i\frac{k^2}{2}s}g_1,$$

and by an approximation argument, for $\eta\in C_0^\infty(\mathbb{R})$,

$$\lim_{t\to\infty}\tilde{U}_0(t)^{-1}h\eta(H_L)\psi_t=\eta\left(\frac{k^2}{2}\right)g_1.$$

This gives $g_1\in L^2((k_2,\sqrt{2L})\times\mathbb{R}^{n-1})$ and thus from (6.19), $\tilde{\Omega}_L^*\psi=g_1$. Finally, we have $\|\tilde{\Omega}_L^*\psi\|=\|g_1\|=\lim_{t\to\infty}\|h\psi_t\|=\|\psi\|$. Since such ψ are dense in $P_{e_1}^{H_L}E_{H_L}\left(\left(\frac{k_2^2}{2},L\right)\right)L^2(\mathbb{R}^n)$, it follows that ker $\tilde{\Omega}_L^*=\{0\}$. □

Appendix: Sternberg linearization with parameters

Suppose $X \in C^\infty(U;\mathbb{R}^n)$ where $U \subset \mathbb{R}^n \times \mathbb{R}^n$ is an open set containing $\{0\} \times U_2$, U_2 open in $\mathbb{R}^n$. We suppose $X(0,k) = 0$ for all $k \in U_2$. We think of $X(\,\cdot\,,k)$ as a vector field depending on the parameter $k \in U_2$. Let $X'(x,k) = D_x X(x,k)$ and denote the eigenvalues of $X'(0,k)$ by $\lambda_i(k)$, $k = 1,\ldots,n$. We suppose that for all $k \in U_2$,

$$\lambda_j(k) - \sum_{\ell=1}^{n} \alpha_\ell \lambda_\ell(k) \neq 0, \text{ all } j \text{ and all } \alpha \text{ with } |\alpha| \geq 2; \tag{A.1}$$

$$\operatorname{Re} \lambda_j(k) < 0, \text{ all } j. \tag{A.2}$$

Here $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $\alpha_\ell \in \{0,1,2,\ldots\}$ all ℓ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Theorem A.1 *Suppose X is as above. Then there exists an open set $V \subset \mathbb{R}^n \times \mathbb{R}^n$ containing $\{0\} \times U_2$ and a C^∞ diffeomorphism $\Psi : V \to \Psi(V) = \tilde{V}$ with $\tilde{V} \subset U$ and Ψ of the form $\Psi(x,k) = (\psi(x,k),k)$ with $\psi(0,k) = 0$, $\psi'(0,k) = I$ such that*

$$\psi'(x,k)X'(0,k)x = X(\psi(x,k),k). \tag{A.3}$$

Remarks. (1) It follows that for fixed k, $\psi(\,\cdot\,,k)$ is a diffeomorphism from its domain onto its image.

(2) Let $\phi_t = \phi_t^{X(\,\cdot\,,k)}$ be the local flow generated by the vector field $X(\,\cdot\,,k)$. Fix k and let B be the domain of $\psi(\,\cdot\,,k)$. Then it is not hard to show that (A.3) is equivalent to the statement,

$$\psi(e^{tX'(0,k)}x,k) \;=\; \phi_t(\psi(x,k)) \text{ for all } (t,x) \text{ such that } e^{sX'(0,k)}x \in B \text{ for } s \text{ in an interval containing } 0 \text{ and } t.$$

(It thus follows from (A.3) that the local flow ϕ_s is defined on $\psi(x,k)$ for s in this interval.)

Our proof of Theorem A.1 follows Nelson's proof of (the easy part of) the Sternberg linearization theorem [N]. We will need the following lemmas.

Lemma A.2 *Suppose $a_\alpha \in C^\infty(U_2)$ for each multi-index $\alpha = (\alpha_1,\ldots,\alpha_n)$. Then there exists a function $f \in C^\infty(\mathbb{R}^n \times U_2)$ such that*

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} f(x,k)|_{x=0} = \alpha!\, a_\alpha(k), \quad k \in U_2. \tag{A.4}$$

Proof. We follow the proof of a standard result in pseudo-differential operators, cf. [Hö, Proposition 18.1.3]. Let $\{\mathcal{O}_n\}_{n=1}^\infty$ be a sequence of open sets with $\overline{\mathcal{O}}_n \subset \mathcal{O}_{n+1}$, $\overline{\mathcal{O}}_n$ compact, and $\bigcup_{n=1}^\infty \mathcal{O}_n = U_2$. Choose $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta = 1$ in a neighborhood of 0 and $\theta(x) = 0$ if $|x| \geq 1$. Choose $\epsilon_0 = 1$ and define $\epsilon_\ell \in (0,1]$ for $\ell \geq 1$ so that

$$\sum_{|\alpha|=\ell} \left| \left(\frac{\partial}{\partial k}\right)^{\beta} \left(\frac{\partial}{\partial x}\right)^{\gamma} \left[a_\alpha(k) x^\alpha \theta\left(\frac{x}{\epsilon_\ell}\right)\right] \right| \leq 2^{-\ell} \text{for } |\beta| \leq \ell - 1,\ |\gamma| \leq \ell - 1,\ k \in \overline{\mathcal{O}}_\ell.$$

This is possible because if $|\alpha| = \ell$ and $|\gamma| < \ell$,

$$\begin{aligned}\left|\left(\frac{\partial}{\partial x}\right)^{\gamma}\left[x^{\alpha}\theta\left(\frac{x}{\epsilon_\ell}\right)\right]\right| &\leq C_\ell \sum_{\gamma_1+\gamma_2=\gamma} |x|^{\ell-|\gamma_1|}\epsilon_\ell^{-|\gamma_2|}\left|\left(\left(\frac{\partial}{\partial x}\right)^{\gamma_2}\theta\right)\left(\frac{x}{\epsilon_\ell}\right)\right| \\ &\leq C'_\ell \epsilon_\ell,\end{aligned}$$

since $|x|^{\ell-|\gamma_1|} \leq \epsilon_\ell^{1+|\gamma_2|}$ if $|x| \leq 1$. We can thus set

$$f(x,k) = \sum_\alpha a_\alpha(k) x^\alpha \theta\left(\frac{x}{\epsilon_{|\alpha|}}\right).$$

If $k \in \mathcal{O}_N$ and $|\beta|$, $|\gamma| \leq N-1$,

$$\sum_{|\alpha|\geq N}\left|\left(\frac{\partial}{\partial k}\right)^{\beta}\left(\frac{\partial}{\partial x}\right)^{\gamma}\left[a_\alpha(k)x^\alpha\theta\left(\frac{x}{\epsilon_{|\alpha|}}\right)\right]\right| \leq \sum_{\ell=N}^{\infty} 2^{-\ell}.$$

Thus $f \in C^\infty(\mathbb{R}^n \times U_2)$ and clearly satisfies (A.4). $\square$

Proposition A.3 *Let $X(x,k)$ be as in Theorem A.1. Then there is a vector field $X_0(x,k)$ with $X_0 \in C^\infty(\mathcal{O})$, $\mathcal{O}$ open, $\{0\} \times U_2 \subset \mathcal{O}$, satisfying*

$$\left(\frac{\partial}{\partial x}\right)^{\alpha}\left[X_0(x,k) - X(x,k)\right]\Big|_{x=0} = 0, \qquad \textit{all } \alpha,\ \textit{all } k \in U_2, \tag{A.5}$$

and a C^∞ diffeomorphism $\Psi_0 : \tilde{\mathcal{O}} \to \mathcal{O}$, where $\Psi_0(x,k) = (\psi_0(x,k),k)$, $\{0\} \times U_2 \subset \tilde{\mathcal{O}}$, and $\psi_0(0,k) = 0$, $\psi_0'(0,k) = I$, such that

$$\psi_0'(x,k)X'(0,k)x = X_0(\psi_0(x,k),k). \tag{A.6}$$

Proof. We first show that we can solve

$$\tilde{\psi}_0'(x,k)X'(0,k)x = X(\tilde{\psi}_0(x,k),k), \tag{A.7}$$

where $\tilde{\psi}_0$ is a formal power series in x with k-dependent coefficients which are C^∞ in U_2. In addition, we demand $\tilde{\psi}_0(0,k) = 0$, $\tilde{\psi}_0'(0,k) = I$. Suppose we have found a polynomial $\psi_{\ell+1}$ of degree ℓ of the form

$$\psi_{\ell+1}(x,k) = x + \sum_{2\leq\alpha\leq l} c_\alpha(k)x^\alpha,$$

with $c_\alpha \in C^\infty(U_2)$ (if $l \geq 2$) so that

$$\psi_{\ell+1}'(x,k)X'(0,k)x = X(\psi_{\ell+1}(x,k),k) + \mathcal{O}(|x|^{\ell+1}). \tag{A.8}$$

Let

$$\psi_{\ell+2}(x,k) = \psi_{\ell+1}(x,k) + \sum_{|\alpha|=\ell+1} c_\alpha(k) x^\alpha,$$

where $c_\alpha(k)$ for $|\alpha| = \ell+1$ is to be determined. We have

$$\begin{aligned}
\psi'_{\ell+2}(x,k)X'(0,k)x - X(\psi_{\ell+2}(x,k),k) &= \psi'_{\ell+1}(x,k)X'(0,k)x - X(\psi_{\ell+1}(x,k),k) \\
&+ \left(\sum_{|\alpha|=\ell+1} c_\alpha(k)x^\alpha\right)' X'(0,k)x \\
&- X'(0,k)\sum_{|\alpha|=\ell+1} c_\alpha(k)x^\alpha + \mathcal{O}(|x|^{\ell+2}).
\end{aligned}$$

From (A.8) we have

$$\psi'_{\ell+1}(x,k)X'(0,k)x - X(\psi_{\ell+1}(x,k),k) = \sum_{|\alpha|=\ell+1} d_\alpha(k)x^\alpha + \mathcal{O}(|x|^{\ell+2}),$$

where $d_\alpha \in C^\infty(U_2)$. Thus we must solve

$$X'(0,k)\sum_{|\alpha|=\ell+1} c_\alpha(k)x^\alpha - \left(\sum_{|\alpha|=\ell+1} c_\alpha(k)x^\alpha\right)' X'(0,k) = \sum_{|\alpha|=\ell+1} d_\alpha(k)x^\alpha.$$

Clearly the map $T(k)$ given by

$$\sum_{|\alpha|=\ell+1} x^\alpha f_\alpha \mapsto X'(0,k)\sum_{|\alpha|=\ell+1} x^\alpha f_\alpha - \left(\sum_{|\alpha|=\ell+1} x^\alpha f_\alpha\right)' X'(0,k)x$$

is a linear map on the finite-dimensional space of $\mathbb{R}^n$-valued homogeneous polynomials of degree $\ell+1$ whose matrix elements (in a k-independent basis) are linear functions of the matrix elements of $X'(0,k)$. The spectrum of $T(k)$ is the set of numbers (see [A] or [N], for example),

$$\left\{\lambda_j(k) - \sum_{\ell=1}^n \alpha_\ell \lambda_\ell(k) : \ |\alpha| = \ell+1, \ j = 1,\ldots,n\right\},$$

which by assumption does not contain 0 if $k \in U_2$. Thus the inverse of $T(k)$ is C^∞ on U_2 and the induction is complete.

It follows that (A.7) can be solved at the level of formal power series.

From Lemma A.2 we can find a function $\hat\psi_0 \in C^\infty(\mathbb{R}^n \times U_2)$ such that

$$\left(\frac{\partial}{\partial x}\right)^\alpha \hat\psi_0(x,k)\Big|_{x=0} = \alpha! c_\alpha(k).$$

Let $\mathcal{O}_n$ be a sequence of open subsets of U_2 with $\overline{\mathcal{O}}_n \subset \mathcal{O}_{n+1}$, $\overline{\mathcal{O}}_n$ compact, and $\bigcup_{n=1}^{\infty} \mathcal{O}_n = U_2$. If $k \in \overline{\mathcal{O}}_n$ we can choose a small open ball B_n centered at 0, independent of k, such that $\hat{\psi}_0(\,\cdot\,, k)|B_n$ is a diffeomorphism onto its image. Let ψ_0 be the restriction of $\hat{\psi}_0$ to $\tilde{\mathcal{O}} = \bigcup_n B_n \times \mathcal{O}_n$ and define $\Psi_0(x,k) = (\psi_0(x,k), k)$. It follows that $\Psi_0 \in C^\infty(\tilde{\mathcal{O}})$ and $\Psi_0 : \tilde{\mathcal{O}} \to \Psi_0(\tilde{\mathcal{O}}) = \mathcal{O}$ is a diffeomorphism. Let

$$Y(x,k) = \psi_0'(x,k) X'(0,k) x.$$

Then from (A.7) it follows that $Y(x,k) - X(\Psi_0(x,k))$ has a Taylor series at $x = 0$ (for fixed $k \in U_2$), which is identically zero. We define

$$X_0(x,k) = Y(\Psi_0^{-1}(x,k)).$$

It is easy to see that (A.5) and (A.6) are satisfied. □

We now construct two vector fields by changing X and X_0 outside a small neighborhood of $\{0\} \times U_2$ to insure they are defined and well behaved for large x. Let $\{\mathcal{O}_n\}_{n=1}^{\infty}$ be a sequence of open subsets of U_2 such that $\overline{\mathcal{O}}_n$ is compact and

$$\mathcal{O}_n \subset \overline{\mathcal{O}}_n \subset \mathcal{O}_{n+1}, \quad \bigcup_{n=1}^{\infty} \mathcal{O}_n = U_2.$$

We can find a sequence of open balls $B_{r_n}(0)$ centered at 0 in $\mathbb{R}^n$ with decreasing radii r_n such that

$$\bigcup_{n=1}^{\infty} B_{r_n}(0) \times \mathcal{O}_n \subset \mathcal{O} \cap U.$$

We construct a positive function $g \in C^\infty(U_2)$ which satisfies

$$\begin{array}{rcll} r_{n+1}^{-1} & \leq & g(k) \leq r_n^{-1} + r_{n+1}^{-1} + r_{n+2}^{-1} & k \in \mathcal{O}_{n+1} \setminus \mathcal{O}_n; \\ r_1^{-1} & \leq & g(k) \leq r_1^{-1} + r_2^{-1} & k \in \mathcal{O}_1. \end{array}$$

This can be done by smoothing out the function

$$g_0(k) = r_1^{-1} \chi_{\mathcal{O}_1}(k) + \sum_{n=1}^{\infty} r_{n+1}^{-1} \chi_{\mathcal{O}_{n+1} \setminus \mathcal{O}_n}(k).$$

Choose $\theta \in C_0^\infty(\mathbb{R}^n)$ with $\theta(x) = 1$ if $|x| \leq \frac{1}{2}$, $\theta(x) = 0$ if $|x| \geq \frac{3}{4}$, and let

$$w(x,k) = \theta(g(k)x) X(x,k) + (1 - \theta(g(k)x)) X'(0,k) x.$$

It is not hard to see that for $k \in \mathcal{O}_{n+1} \setminus \mathcal{O}_n$,

$$\|w'(x,k) - X'(0,k)\| \leq \frac{c_n}{g(k)} \leq r_{n+1} c_n, \quad \text{all } x \in \mathbb{R}^n,$$

where c_n is independent of the r_n's. Thus by choosing the r_n's sufficiently small we can make sure that for $k \in \mathcal{O}_n$,

$$|\phi_t^{w(\,\cdot\,,k)}(x)| \leq d_n e^{-\gamma_n t}|x|, \quad t \geq 0,\ \gamma_n > 0. \tag{A.9}$$

Here ϕ_t^w is the global flow generated by w. Similarly, defining

$$v(x,k) = \theta(g(k)x)X_0(x,k) + (1-\theta(g(k)x))X'(0,k)x,$$

we can assume

$$|\phi_t^{v(\,\cdot\,,k)}(x)| \leq d_n e^{-\gamma_n t}|x|, \quad t \geq 0,\ \gamma_n > 0. \tag{A.10}$$

Proposition A.4 *Let $v(x,k)$ and $w(x,k)$ be as above. There exists a C^∞ map Ω : $\mathbb{R}^n \times U_2 \to \mathbb{R}^n$ such that $\Omega(\,\cdot\,,k)$ is a diffeomorphism $\Omega(\,\cdot\,,k):\ \mathbb{R}^n \overset{\text{onto}}{\to} \mathbb{R}^n$ for each k and*

$$\Omega(0,k) = 0, \quad \Omega'(0,k) = I;$$

$$\Omega'(x,k)v(x,k) = w(\Omega(x,k),k).$$

Proof. We follow Nelson [N] and use wave operators to construct Ω. For brevity we omit the explicit dependence on k and, for example, write $v(x,k) = v(x)$, etc. We use the linear operator e^{tv} given by

$$e^{tv}f(x) = f(\phi_t^v(x)).$$

It follows that $\frac{d}{dt}e^{tv} = ve^{tv} = e^{tv}v$ and thus by integrating the derivative

$$e^{tv}e^{-tw} = I + \int_0^t e^{sv}(v-w)e^{-sw}ds. \tag{A.11}$$

Applying (A.11) to the function $f(x) = x$ we obtain

$$\phi_{-t}^w \circ \phi_t^v(x) = x + \int_0^t (\phi_{-s}^w)'(\phi_s^v(x))z(\phi_s^v(x))ds, \tag{A.12}$$

where $z(x) = v(x) - w(x)$. Suppose $k \in \mathcal{O}_n$, so that

$$|\phi_t^w(x)| \leq de^{-\gamma t}|x|, \quad |\phi_t^v(x)| \leq de^{-\gamma t}|x|, \tag{A.13}$$

$$|w'(x)| \leq \kappa, \quad |v'(x)| \leq \kappa. \tag{A.14}$$

All our estimates will be uniform for $k \in \mathcal{O}_n$. We omit the n-dependence of constants like d, γ, κ. Using the differential equation of the flow ϕ_t^w, we obtain for $t \geq 0$,

$$\begin{aligned}(\phi_t^w)'(x) &= Te^{\int_0^t w'(\phi_s^w(x))ds};\\ (\phi_{-t}^w)'(x) &= Te^{-\int_0^t w'(\phi_{-s}^w(x))ds},\end{aligned}$$

where T indicates the time-ordered product integral (see [N] for example). Thus, for $t \geq 0$,

$$\|(\phi^w_{-t})'(x)\| \leq e^{\kappa t}.$$

$v(x)-w(x)$ has compact support and vanishes to infinite order at $x=0$. Thus, choosing m so that $m\gamma > \kappa$ we estimate

$$\begin{aligned}\int_t^\infty |(\phi^w_{-s})'(\phi^v_s(x))z(\phi^v_s(x))|ds &\leq \int_t^\infty e^{\kappa s}(de^{-\gamma s})^m|x|^m ds \\ &= d^m(m\gamma-\kappa)^{-1}|x|^m e^{-(m\gamma-\kappa)t}.\end{aligned}$$

It follows that

$$\lim_{t\to\infty}\phi^w_{-t}\circ\phi^v_t(x) = \Omega_{w,v}(x)$$

exists uniformly on compacts of $\mathbb{R}^n \times U_2$ and thus defines a continuous function. Similarly, we find $\Omega_{v,w} \in C(\mathbb{R}^n \times U_2)$. It is not difficult to show that for fixed k,

$$\begin{aligned}\Omega_{w,v}\circ\Omega_{v,w} &= \Omega_{v,w}\circ\Omega_{w,v} = I; \\ \Omega_{w,v}\circ\phi^v_s &= \phi^w_s\circ\Omega_{w,v}.\end{aligned} \tag{A.15}$$

Thus, $\Omega_{w,v}$ is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which is jointly continuous in (x,k).

We let $\Omega = \Omega_{w,v}$. From (A.15) we obtain

$$\Omega(x) = \phi^w_{-t}(\Omega(\phi^v_t(x))).$$

It follows that any order of differentiability of Ω in (x,k) for small x implies the same for large x.

To prove differentiability in k, let $D_k = a\cdot\nabla_k$ for some $a \in \mathbb{R}^n$ and consider the flow $\Phi^{\tilde{v}}_t$ on $\mathbb{R}^n \times \mathbb{R}^n$ generated by the vector field

$$\tilde{v}(x,\xi) = (v(x), D_k v(x) + v'(x)\xi).$$

Note that

$$\tilde{v}'(0,0) = \begin{pmatrix} v'(0) & 0 \\ D_k v'(0) & v'(0)\end{pmatrix}$$

has the same spectrum as $v'(0)$ (with higher multiplicity), and that if

$$\tilde{w}(x,\xi) = (w(x), D_k w(x) + w'(x)\xi), \tag{A.16}$$

$\tilde{v}$ and $\tilde{w}$ have the same Taylor series around $(x,\xi) = (0,0)$. The vector field $\tilde{v}$ arises when we look at the evolution of $D_k\phi^v_t(x)$, which satisfies the differential equation

$$\frac{d}{dt}D_k\phi^v_t(x) = D_k v(\phi^v_t(x)) + v'(\phi^v_t(x))D_k\phi^v_t(x).$$

Evidently,

$$\Phi_t^{\tilde{v}}(x,0) = (\phi_t^v(x), D_k\phi_t^v(x)). \tag{A.17}$$

We claim

$$\Phi_{-t}^{\tilde{w}} \circ \Phi_t^{\tilde{v}}(x,0) = (\phi_{-t}^w \circ \phi_t^v(x), D_k(\phi_{-t}^w \circ \phi_t^v(x))). \tag{A.18}$$

To prove (A.18), let $(x(s), y(s)) = \Phi_{-s}^{\tilde{w}} \circ \Phi_t^{\tilde{v}}(x_0, 0)$. Clearly, $x(s) = \phi_{-s}^w \circ \phi_t^v(x_0)$. We have (see (A.16) and (A.17))

$$\frac{dy(s)}{ds} = -[D_k w(x(s)) + w'(x(s))y(s)], \quad y(0) = D_k\phi_t^v(x_0). \tag{A.19}$$

On the other hand,

$$\begin{aligned} \frac{d}{ds} D_k x(s) &= D_k \frac{d}{ds} x(s) = -D_k[w(x(s))] \\ &= -((D_k w)(x(s)) + w'(x(s))D_k x(s)), \; D_k x(0) = D_k\phi_t^v(x_0).) \end{aligned} \tag{A.20}$$

Comparing (A.19) and (A.20) we have $y(s) = D_k x(s)$ which gives (A.18). We claim that $\Phi_{-t}^{\tilde{w}} \circ \Phi_t^{\tilde{v}}(x,\xi)$ converges uniformly for (x,ξ) small and k in a compact set. To see this we use the same procedure as previously. We modify the vector fields $\tilde{w}$ and $\tilde{v}$ outside a neighborhood of $(x,\xi) = (0,0)$ and estimate the analog of (A.12) as before. We claim that the limit we obtain after modification is equal to $\lim_{t\to\infty} \Phi_{-t}^{\tilde{w}} \circ \Phi_t^{\tilde{v}}(x,\xi)$ for (x,ξ) sufficiently small. Thus consider (A.12) for x small. The argument of z can be made as small as desired for all $s \in [0,\infty)$ by taking x sufficiently small and thus $z(\phi_s^v(x))$ does not change if x is sufficiently small. We have

$$(\phi_{-s}^w)'(\phi_s^v(x)) = Te^{-\int_0^s w'(\phi_{-\tau}^w(\phi_s^v(x)))d\tau}.$$

Thus we must show that $\phi_{-s}^w \circ \phi_t^v(x)$ can be made as small as desired for all s, t with $0 \le s \le t$ by taking x sufficiently small. Estimating (A.12) we obtain

$$|\phi_{-t}^w \circ \phi_t^v(x) - x| \le d^m(\gamma m - \kappa)^{-1}|x|^m, \tag{A.21}$$

while $|\phi_{s-t}^w \circ \phi_t^v(x)| \le ce^{-\gamma s}|\phi_{-t}^w \circ \phi_t^v(x)|$, for $s \in [0,t]$, which proves our claim. It follows that $D_k(\Omega)$ is continuous in all variables. A similar argument (see [N]) applies to $D_x\phi_{-t}^w \circ \phi_t^v(x)$. An induction argument then shows that $\Omega = \Omega_{w,v}$ is C^∞ in all variables. A similar argument also applies to $\Omega^{-1} = \Omega_{v,w}$. Thus $\Omega(\,\cdot\,,k)$ is a diffeomorphism for each k, and the rest of the Proposition follows from (A.15) and (A.21). □

Proof of Theorem A.1. We restrict the domain of $\Omega(x,k)$ to a small enough open set U_1 containing $\{0\} \times U_2$ but contained in $\mathcal{O}$ so that the restriction $\tilde{\Omega} = \Omega|U_1$ satisfies

$$\tilde{\Omega}'(x,k)X_0(x,k) = X(\tilde{\Omega}(x,k),k).$$

In combination with Proposition A.3 it follows that

$$\begin{aligned} \tilde{\Omega}(\psi_0(\cdot,k),k)'X'(0,k)x &= \tilde{\Omega}'(\psi_0(x,k),k)X_0(\psi_0(x,k),k) \\ &= X(\tilde{\Omega}(\psi_0(x,k),k),k), \end{aligned}$$

or if $\psi(x,k) = \tilde{\Omega}(\psi_0(x,k),k)$,

$$\psi'(x,k)X'(0,k)x = X(\psi(x,k),k),$$

which is (A.3). If we let $\Gamma(x,k) = (\tilde{\Omega}(x,k),k)$, $\Psi_0(x,k) = (\psi_0(x,k),k)$, and define $\Psi = \Gamma \circ \Psi_0$, then $\Psi(x,k) = (\psi(x,k),k)$. The domain of Ψ is $V = \Psi_0^{-1}(U_1)$. The fact that $\psi(0,k) = 0$ and $\psi'(0,k) = I$ is clear. $\square$

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