SOME FEATURES OF VIBRATIONS IN HOMOGENEOUS 1D RESISTANT ELASTIC LUMPED LINE

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This paper is one of a large cycle of papers on obtaining the exact analytical solutions for vibrant elastic systems. This was published first in the electronic journal of laboratory SELF "SELF Transactions", vol.2, issue 1, pp.17-34. Visiting this journal for free at
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you will find many other important results both on this subject and on many applications of this new method, as well as on many other subjects of mathematical physics and related themes, such as quasi-conformal mapping, vector algebra, elasticity, acoustics, circuit theory, dynamical electromechanical analogy, electric mismatched ladder filters, transmission lines, networks, field theory, Maxwell equations, geophysics. You will also find there that we check our results experimentally and yield the full coincidence of the calculation and experimental curves. And the main, this method gives exact solutions, and this is non-matrix. Already now we have solved many practically important problems which cannot be even approached properly by conventional methods. This is the future of dynamics out of which any development in any area of basic or applied wave physics is impossible. You are welcomed to possess this wealth and contact us for discussions and joint further development.

Abstract

In this paper we will consider the effect of resistance on vibration processes in a semi-infinite elastic lumped and distributed lines. We will see that for the given type of line, the progressive pattern of vibrations remains also at the overcritical frequencies, and the phase delay is always less than the value, corresponding to the first Brillouin zone. Comparing the obtained results with experimental data of the phase velocity of ultrasonic wave in the carbonic acid gas, we will see that taking the resistance into consideration, we can essentially refine the conventional models and promote their better correspondence to the experimental data.

Keywords: Many-body theory, Wave physics, Complex resonance systems, ODE

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1. Introduction

The most of applied models consider the dissipative force of resistance. Many works, particularly [1] – [9], have been devoted to this subject. However, the investigation is hampered by the influence of a few factors. First, the conventional mathematical methods for elastic lumped lines calculation lead, as a rule, either to the matrix methods that for multibody systems are able to be solved only numerically, or immediately to the numerical methods like Ritz method, asymptotic methods and some others. For example, in [8], for a conventional modelling system basing on Lagrange equation

$$\sum_{k=1}^{s}\left\{a_{jk}\ddot{\xi}_k+\left(b_{jk}-\gamma_{jk}\right)\dot{\xi}_k+c_{jk}\xi_k\right\}=Q_j^e\left(t\right) \tag{1}$$

(here $Q_j^e\left(t\right)$ are the given functions of time, a_{jk}, b_{jk}, c_{jk}, γ_{jk} are the parameters characterising the elastic system, ξ_k are the generalised co-ordinates of the kth element of a system, and s is the number of system elements), the complex vibration amplitude is sought in the form [8, p.296]

$$a_k^e\, e^{j\Delta\beta_k^e}=\frac{\Delta^k\left(\lambda_e\right)}{\Delta\left(\lambda_e\right)} \tag{2}$$

(here λ_e is the wavelength corresponding to the external action frequency, and $\Delta\beta_k^e$ is the damping factor in the system). In (2), $\Delta\left(\lambda_e\right)$ is the characteristic determinant of (1) taken at $\lambda=\lambda_e$, and $\Delta^k\left(\lambda_e\right)$ is the determinant obtained from that characteristic by replacing the kth column by a column composed of Q_1^e, Q_2^e, ... Q_s^e. With s growing, these determinants practically can have only numerical solution, which naturally hampers the analysis of resistance effect on a system.

Second, the solutions that one can obtain by a conventional method are quite incomplete. As [10] – [13] show, the conventional methods do not give one a scope to study vibrations in an overcritical domain (over the boundary frequency ω_0). "In the vibration theory, the solution of boundary vibrations reduces basically to the eigenvalues determination connected with the natural frequencies or other parameters of the studied system and to finding the eigenfunctions (of the vibration forms). If the eigenvalues and eigenfunctions have been found, then we can think the boundary problem solved... At present a great number of approximate methods has been developed for finding the eigenvalues, but they all are quite laborious, give only the first eigenvalues, and the main, do not unite the studying of systems with discrete and continuous mass distribution" [5, pp.3-4]. Only in some particular cases we have the incomplete, limitedly applicable solutions or mentionings that the processes in the overcritical domain are possible (see, e.g., [7, pp.282 – 285], [8, p.294], [14, p.109]). At the same time, it is inherent in the vibration process in this domain that at the frequencies higher than critical, even an ideal elastic line behaves as a natural damper with the along-the-line damping vibration amplitude. As we will show in this paper, the resistance essentially effects on the damping characteristics, as well as on the vibration pattern of the system elements. To make the analysis complete and high-quality, it is important to take into account, how the resistance effects on the elastic system vibrations. In the course of present investigation we will limit ourselves intentionally with a simple model of a semi-infinite 1D elastic line, to visualise the analysed processes. We will suppose that for other types of elastic lines we can obtain the alike results, since now we have available such exact analytical solutions as have been presented in [10] – [13].

2. Exact analytical solutions for a semi-infinite 1D elastic resistant lumped line

Seeking the exact analytical solutions for a studied elastic line, we will choose not a conventional way to find the particular solutions for a modelling system of differential equations. We will try to reduce the problem to the solutions for an ideal line presented in [10], the more it will be simple.

In our case the modelling system of differential equations will take the form

$$m\frac{d^2\Delta_1}{dt^2} = F(t) - r\frac{d\Delta_1}{dt} + s(\Delta_2 - \Delta_1)$$
$$m\frac{d^2\Delta_2}{dt^2} = -\, r\frac{d\Delta_2}{dt} + s(\Delta_3 + \Delta_1 - 2\Delta_2)$$
$$\dots\dots\dots\dots\dots\dots \qquad (3)$$
$$m\frac{d^2\Delta_n}{dt^2} = -\, r\frac{d\Delta_n}{dt} + s(\Delta_{n+1} + \Delta_{n-1} - 2\Delta_n)$$
$$\dots\dots\dots\dots\dots\dots,$$

where Δ_n is the nth line element displacement, and $F(t) = F_0 e^{j(\omega t - \varphi_0)}$. Here and further $n = 1, 2, 3, \dots$ and s is the stiffness coefficient of a line. After the conventional elimination the time dependence

$$\Delta_n = \delta_n e^{j(\omega t - \varphi_0)} \qquad (4)$$

and substitution

$$m' = \left(m - j\frac{r}{\omega}\right) \qquad (5)$$

the system (3) takes the form

$$-\omega^2 m'\delta_1 = F_0 + s(\delta_2 - \delta_1)$$
$$-\omega^2 m'\delta_2 = s(\delta_3 + \delta_1 - 2\delta_2)$$
$$\dots\dots\dots\dots\dots\dots \qquad (6)$$
$$-\omega^2 m'\delta_n = s(\delta_{n+1} + \delta_{n-1} - 2\delta_n)$$
$$\dots\dots\dots\dots\dots\dots\ .$$

We can easy see that (6) is fully identical to the conventional modelling system of differential equations for an ideal semi-infinite elastic line, for which in [10] the exact analytical time-dependent solutions of the following form have been obtained:
for the *periodical* regime $(\omega > \omega_0, \beta > 1)$

$$\Delta_n = -j\frac{F_0}{\omega\sqrt{sm'}} e^{-j[\omega t - (2n-1)\tau + \varphi_0]}, \qquad (7)$$

for the *aperiodical* regime $(\omega > \omega_0, \beta > 1)$

$$\Delta_n = (-1)^n \frac{F_0}{2s\beta}\gamma_-^{2n-1} e^{j(\omega t + \varphi_0)}, \qquad (8)$$

and for the *critical* regime $(\omega = \omega_0, \quad \beta = 1)$

$$\Delta_n = (-1)^n \frac{F_0}{2s} e^{j(\omega t + \varphi_0)}, \qquad (9)$$

where

$$\beta = \sqrt{\frac{\omega^2 m'}{4s}}, \qquad (10)$$
$$\tau = \arcsin\beta, \qquad (11)$$
$$\gamma_- = \beta - \sqrt{\beta^2 - 1}. \qquad (12)$$

Applying (7) – (9) to (6) as the solutions, we should take into account that after we substituted (5), parameter β in (10) becomes a complex value:

$$\beta = \sqrt{\frac{\omega^2 m'}{4s}} = \sqrt{\frac{\omega^2}{4s}\left(m - j\,\frac{r}{\omega}\right)} = \frac{1}{2\sqrt{2s}}\left[\sqrt{\sqrt{\omega^4 m^2 + r^2\omega^2} + \omega^2 m} - \right.$$
$$\left. - j\sqrt{\sqrt{\omega^4 m^2 + r^2\omega^2} - \omega^2 m}\right]. \tag{13}$$

It means that in a resistant line only the periodical regime (7) can exist. The critical and aperiodical regimes are possible only when transiting to an ideal line, i.e. at a real value β realisable at $r = 0$, as we can see it in (13).

It is convenient to transform (7) into an algebraic form, noting (11):

$$\Delta_n = -j\frac{F_0\, e^{\,j(\omega t + \varphi_0)}}{\omega\sqrt{sm'}}\left(\sqrt{1-\beta^2} - j\beta\right)^{2n-1}. \tag{14}$$

Now substituting (5) and (13) into (14), after a transformation we obtain the sought solution in the form

$$\Delta_n = -j\frac{F_0\, e^{j(\omega t + \varphi_0)}\, e^{j\varphi_{0r}}}{\sqrt{s}\left(\omega^4 m^2 + r^2\omega^2\right)^{1/4}} R^{2n-1} e^{-j(2n-1)\varphi_r}, \tag{15}$$

where

$$R = \left[A + C - \sqrt{(C+D)(A-B)} - \sqrt{(C-D)(A+B)}\right]^{1/2}, \tag{16}$$

$$\varphi_r = \arcsin\frac{\sqrt{A+B} - \sqrt{C-D}}{\sqrt{2}R}, \tag{17}$$

$$\varphi_{0r} = \arcsin\frac{\sqrt{A-B}}{\sqrt{2A}}, \tag{18}$$

$$A = \frac{1}{4s}\sqrt{\omega^4 m^2 + r^2\omega^2}, \tag{19}$$

$$B = \frac{m\omega^2}{4s}, \tag{20}$$

$$C = \sqrt{\left(1 - \frac{m\omega^2}{4s}\right)^2 + \frac{r^2\omega^2}{16s^2}}, \tag{21}$$

$$D = 1 - \frac{m\omega^2}{4s}. \tag{22}$$

Comparing (15) with (14), we see that the resistance reflects on the solution in two ways. It effects on the vibration pattern on the whole and on the link-to-link excitation transmission. The multiplier $\left(\omega^4 m^2 + r^2\omega^2\right)^{1/4}$ effects on the amplitude as a whole. Its influence is especially considerable at the low and ultralow frequencies, when $r^2\omega^2$ is comparable with $\omega^4 m^2$. It is quite limited band, out of which in practical calculations we can neglect $r^2\omega^2$ in comparison with $m^2\omega^4$.

The parameter φ_{0r} also effects on the vibration phase on the whole. According to (18) and taking into account (19) and (20), this influence is also limited by low and ultralow frequencies. At $r << m\omega$, the vibration phase φ_{0r} can be approximately calculated as

$$\varphi_{0r} \approx \frac{r}{2m\omega}. \tag{23}$$

Despite so limited influence on these parameters, we have to take them into consideration when studying the ultralow-band vibrations and broad-range excitations, since the presence of φ_{0r} leads

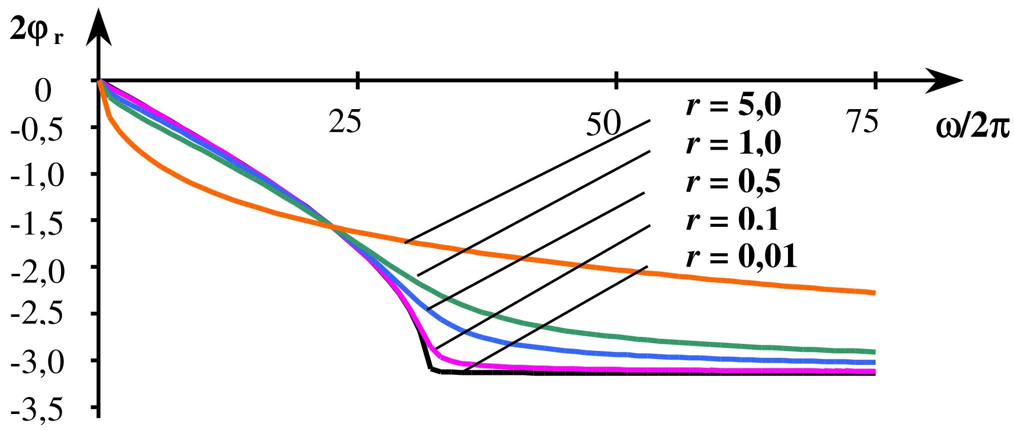

Fig.1. The phase delay $2\varphi_r$ produced by an elastic lumped line section with respect to the external action frequency ω, at different values of the section resistance *r*

to the dependence of the initial phase of vibration process on the frequency, because of hyperbolic dependence of this phase on frequency in (23).

3. Analysis of excitation transmission process in a resistant elastic line

The parameters R and φ_r have the main effect on the resistant elastic line vibrations. To analyse them, conveniently use the transfer function of a link G_r by the analogy with electric filters:

$$G_r = \frac{\Delta_{n+1}}{\Delta_n} = R^2 e^{-j2\varphi_r}. \tag{24}$$

According to (24), the transmission coefficient of each link is proportional to R^2 and the delay caused by each link – to $2\varphi_r$. This process is inherent in all the frequency range from zero to infinity. At the same time, according to (7) – (9), the transfer function of a link G_0 for an ideal line is different in all the three bands of β variation:

$$\begin{cases} G_0 = e^{-j2\tau}; \quad \beta < 1 \\ G_0 = e^{-j\pi}; \quad \beta = 1 \\ G_0 = \left(\beta - \sqrt{\beta^2 - 1}\right) e^{-j\pi}; \quad \beta > 1. \end{cases} \tag{25}$$

In accordance with (11) and (17), both parameters – φ_r and τ – depend on frequency nonlinearly. At $r \to 0$ we have $\varphi_r \to \tau$, $R \to 1$ and $G_r \to G_0$. It means that an ideal line reflects the processes occurring in real lines as some approximate model, despite in the general solutions three frequency bands appear. The plots $2\varphi_r(\omega)$ and $R^2(\omega)$ shown in Fig. 1 and Fig. 2 corroborate it. Despite in a resistant line only the periodical vibration regime can be realised, in the plots we clearly see that at small r at the sub-critical band, $2\varphi_r$ varies as arc sine and the damping is absent, because $R = 1$. At the overcritical band, the phase is constant and equal practically to π, but the damping arises. The antiphase vibration damping at the neighbouring elements of a line cause it. At the point of critical regime, which in Fig. 1 and Fig. 2 corresponds to the frequency $f_0 = 31{,}8$ Hz, the kink of characteristics occurs. In other words, at small r a resistant line exactly repeats the features of the periodical, critical and aperiodical regimes in a corresponding ideal line. The only distinction is, for a resistant line all the features of vibration regimes are described by the unified analytical functions, not a system like (25). When calculating the elastic systems having small r, it

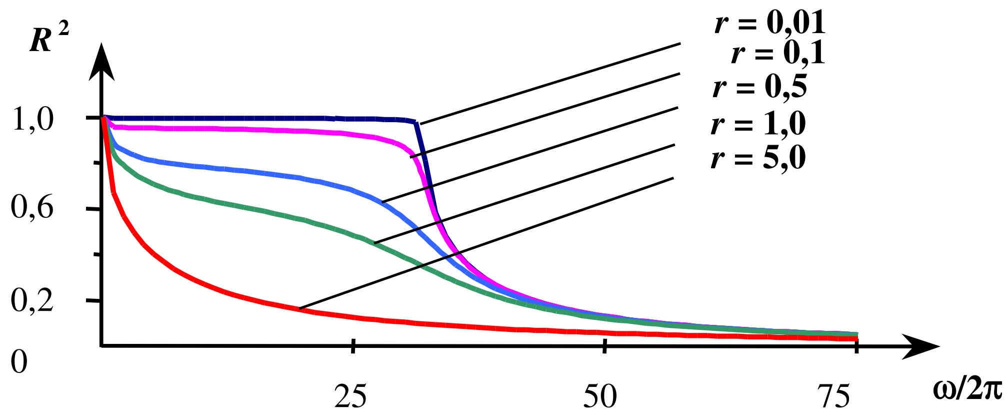

Fig. 2. The attenuation degree R^2 created by an elastic lumped line section, with respect to the external action frequency ω, at different values of the line section resistance r

enables using the ideal line model in which the conditions causing the line to pass or stop vibtations have been differentiated more clearly for a wave process.

With the resistance rising, at the subcritical band the plot $2\varphi_r(\omega)$ differs from arc sine more and more. At the overcritical band $2\varphi_r$ also is not equal to π, but asymptotically tends to this value. This is why the aperiodical regime is impossible in a resistant line. However, the worsening of damping peculiarities caused by the disturbance of antiphase vibrations of the neighbouring elements does not diminish the damping introduced by $R^2(\omega)$. It even increases, when the dissipative peculiarities of a line grow. The more the resistance r grows the more the damping has an effect on the subcritical band, and the transition being typical for the critical regime smoothes, leading $R^2(\omega)$ to the asymptotic form at large r. We can see the similar transformations also dependently on $2\varphi_r(\omega)$. The transition typical for the critical regime also smoothes in it, and at large r the regularity $2\varphi_r(\omega)$ takes a form of a smooth function. At the same time, in case of neither ideal nor resistant line, the phase delay $2\varphi_r$ exceeds π, and at $r \neq 0$ reaches it on all the range. This result refines the conventional concept that at the overcritical regime the neighbouring elements vibration phase exceeds π, "so it is senseless to describe the phase displacements exceeding 180° " [14, p.110]. Furthermore, at present there takes place a view that “in this connection, the values $\bar{k}$ for which $|\bar{k}| > \dfrac{\pi}{a}$ have not a physical meaning” [14, p.109], and “the waves with the angular velocity more than $\omega_m = 2v_0/a$ cannot exist in an imaginary one-dimensional crystal… Their wave vector, whose imaginary component causes a strong absorption, has to be complex. Thus, the frequencies for which the inequality $\omega > \omega_m$ is true appear at the stop band” [14, p.110]. The same in [15, p.166]: "Since a wave has physical meaning only at the lattice points n, the wave having the argument $\tau' = \tau + \pi/a$ describes just the same process as the wave with the argument τ, so we will consider only the waves (in both directions) whose length $\lambda > 4a$" (here $2a$ is the lattice constant). None the less, the above consideration shows that at all the range (not only at the subcritical band) the phase delay does not exceed the limits of the first Brillouin zone determined as $|\bar{k}| = \dfrac{\pi}{a}$ or $2\varphi_r = \pi$. The more, basing on the solutions obtained above, we can show that not a standing wave, as one used to think, but a progressive wave transferring the energy along the line corresponds to

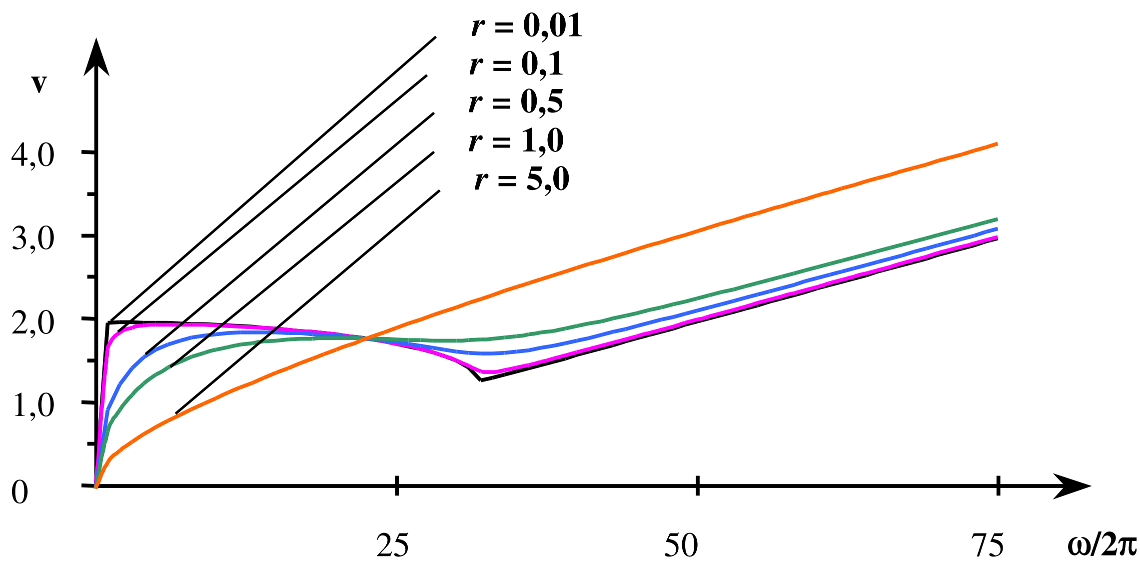

Fig. 3. The phase velocity v of wave propagation in a lumped line with respect to the external action frequency ω, at different values of the line section resistance r

the overcritical band of vibrations. To make it sure, determine sequentially the dependencies of the phase velocity v_{ph} and group velocity v_{gr} on the frequency.

To find v_{ph}, note that the wavelength λ in an elastic lumped line can be presented as

$$\lambda = \frac{2\pi a}{2\varphi_r}, \tag{26}$$

where a is the distance between the unexcited elements of line. With (26), the conventional expression for the phase velocity will take the form

$$v_f = \frac{\omega}{k} = \frac{a\omega}{2\varphi_r}. \tag{27}$$

Substituting (17) into (27), we yield the sought regularity whose pattern is presented in Fig. 3. We see that in the subcritical band at small r the phase velocity in the beginning of frequency band increases rapidly from zero to some local maximum. After it, the velocity slowly falls, and after the critical frequency it grows again practically linearly. It evidences that in a resistant line, in all the range, the progressive pattern of wave propagation takes place. At the low-frequency and overcritical bands a normal (positive) phase velocity dispersion takes place. While at the intermediate band, an anomalous (negative) dispersion occurs. With growing resistance, the anomalous dispersion band narrows, and at large r the dependence $v_{ph}(\omega)$ becomes a monotonously growing function characterising the normal dispersion at all the range.

Considering an ideal elastic line as some limiting case of a resistant line, we can state that in this model in the aperiodical regime also, the progressive wave will propagate, and its length will stabilise at $\lambda = 2a$. We can judge it, particularly, by a limiting curve to which the plots $v_{ph}(\omega)$ in Fig. 3 are tighten at $r \to 0$. With it we have to note the peculiarities that appear in the solutions as a result of idealisation. Particularly, in Fig. 3 we see that when the frequency tends to zero, the phase velocity also tends to zero. It will occur otherwise in ideal lines, and the phase velocity will tend to the conventional value $\sqrt{T_l/\rho}$ at $\omega \to 0$. It is easy to check it, substituting the value τ from (11) to (27) instead φ_r and finding the limit at $\omega \to 0$. None the less, as we will see below, the real-world

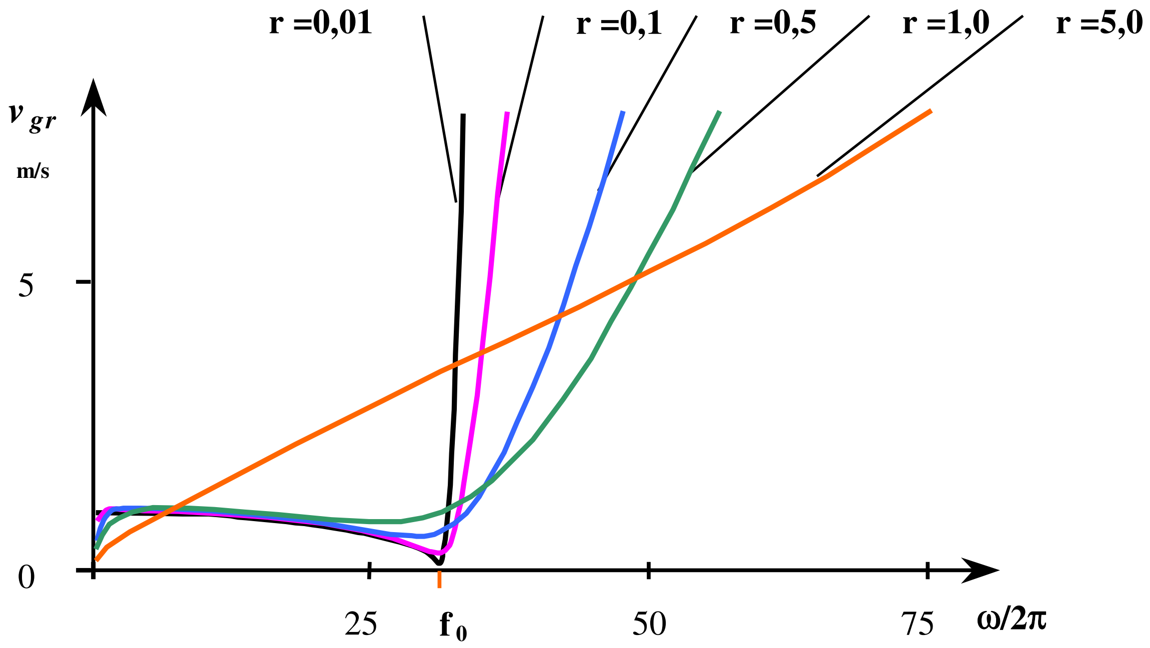

Fig. 4. The group velocity with respect to frequency at different values of the elastic line resistance

systems have some features of solutions, which one cannot obtain on the idealised models; however it is desirable to take them into consideration, when idealising the real systems.

The same will be for the group velocity of wave propagation. As we know (see e.g. [6], [9], [14]), at present, one associates the energy transfer by a wave with this velocity, considering that "when the neighbouring atoms move in antiphase, which is realisable for $\left|\bar{k}\right| = \pm\dfrac{\pi}{a}$, i.e. for the wavelength $\lambda = 2a$, the group velocity appears zero" [14, p.109]. Due to it, in the overcritical band the along-the-line energy transfer is absent, since "the bound of Brillouin zone $\left|\bar{k}\right| = \dfrac{\pi}{a}$ will correspond not to a progressive but to a standing wave" [14, p.110].

To plot the group velocity with respect to frequency $v_{gr}(\omega)$ in case of elastic lumped line, we have to pay attention that not the phase delay φ_r but ω is an independent variable in (17). So

$$v_{gr} = \frac{\partial\omega}{\partial k} = -\frac{a}{2}\frac{1}{\partial\varphi_r/\partial\omega}. \qquad (28)$$

Finding the derivative $\partial\varphi_r/\partial\omega$ from (17) and substituting its value to (28), we determine the group velocity of wave propagation with respect to frequency. A typical plot of this regularity is shown in Fig. 4. First of all we see the alike patterns of regularity $v_{ph}(\omega)$ in Fig. 3 and $v_{gr}(\omega)$ in Fig. 4 at the subcritical band and small r. In both plots we see an abruptly increasing velocity at the ultralow frequency band followed by the anomalous dispersion up to the boundary frequency ω_0. At $\omega = \omega_0$ the phase velocity takes some finite value, while the group velocity in case of an ideal line vanishes. *But only for an ideal, not resistant line*. At the boundary frequency, the more resistance is the more group velocity differs from zero.

At the overcritical band, the more group velocity increases the less resistance r is. But despite the abrupt increase, its value stays finite. Hence, for all resistant elastic lines (even at an infinitesimal resistance), at the overcritical band the along-the-line energy propagation takes place, though with a large amplitude damping.

In a limiting case of an ideal elastic line, the group velocity at the overcritical band vanishes, and we have to take this feature into account when analysing the results obtained with a definite idealisation of the modelled process.

At the large resistance, $v_{gr}(\omega)$ becomes a monotonously growing function of frequency having no features at the point of critical regime. The pattern of its dependence on frequency is similar to $v_{ph}(\omega)$ at large r.

4. Limit passing to a distributed line

The exact analytical solutions (19) obtained above for a lumped line can be easy extended to that distributed. To do so, we have to substitute the parameters characterising the discrete mass distribution for a corresponding system of parameters characterising 1D system of distributed masses. Introduce

$$\rho=\frac{m}{a};\quad n=\frac{x_0}{a};\quad \sigma=\frac{r}{a};\quad T_l=sa;\quad x-x_0=\Delta_n, \tag{29}$$

where ρ is the elastic line density, x is the momentary co-ordinate of the studied point of a line in the state of rest, x_0 is the studied point co-ordinate in the state of rest, T_l is the line stiffness, σ is the mechanical specific resistance of a line, a is the distance between the elements of a lumped line.

Having available (29), we can determine the transformation of parameters (16) – (18) at $a\to 0$. First of all, at small a, the parameters A, B, C, D can be written down, noting (29):

$$\begin{gathered}A=\frac{a^2}{4T_l}\sqrt{\rho^2\omega^4+\sigma^2\omega^2};\quad B=\frac{a^2\rho\omega^2}{4T_l};\\ C\simeq 1-\frac{\rho\omega^2a^2}{4T_l};\quad D=1-\frac{\rho\omega^2a^2}{4T_l}.\end{gathered} \tag{30}$$

Substituting (30) into (16), we obtain accurate to a:

$$\begin{gathered}R\approx\left[1-\frac{a}{2\sqrt{T_l}}\sqrt{\sqrt{\rho^2\omega^4+\sigma^2\omega^2}-\rho\omega^2}\right]^{1/2}=\left[1-a\chi(\rho,\sigma,\omega,T)\right]^{1/2}\approx\\ \approx\left[1+a\chi(\rho,\sigma,\omega,T)\right]^{-1/2},\end{gathered} \tag{31}$$

where $\chi=\frac{1}{2\sqrt{T_l}}\sqrt{\sqrt{\rho^2\omega^4+\sigma^2\omega^2}-\rho\omega^2}$ contains only the values corresponding to a distributed system.

In further studying the parameter R, we have to note its power in (15). Then from (29) and (31) we can write

$$\lim_{a\to 0}R^{2n-1}=\lim_{a\to 0}\left[1+a\chi\right]^{-\frac{1}{2}\left(\frac{2x_0}{a}-1\right)}=\lim_{a\to 0}\left[\left(1+a\chi\right)^{\frac{1}{a\chi}}\right]^{-\chi x_0}=e^{-\chi x_0}. \tag{32}$$

It means, when transiting to a distributed line, the along-the-line damping is described by an exponential regularity. When transiting to an ideal line, i.e. at $\sigma\to 0$, according to (31), χ also tends to zero, as it was expected. The dependence $\chi(\omega)$ reveals especially at low frequencies, when σ/ω is comparable with ρ. At high frequencies, the expression for χ can be presented in an approximate form

$$\chi\approx\frac{1}{2\sqrt{2T_l}}\frac{\sigma}{\omega\rho}\ . \tag{33}$$

The parameter φ_r at the limit passing has to be considered also jointly with the coefficient $(2n-1)$. Noting (17), (29) and (30), we yield

$$\lim_{a\to 0}(2n-1)\varphi_r = \lim_{a\to 0}\left(\frac{2x_0}{a}-1\right)\frac{a\sqrt{\sqrt{\rho^2\omega^4+\sigma^2\omega^2}+\rho\omega^2}}{2\sqrt{2T_l}R} = \\ =\omega x_0\sqrt{\frac{\sqrt{\rho^2+\sigma^2/\omega^2}+\rho}{2T_l}} = k_r x_0 , \tag{34}$$

where $k_r = \omega\sqrt{\frac{\sqrt{\rho^2+\sigma^2/\omega^2}+\rho}{2T_l}}$ is the wave coefficient of a resistant line.

It is characteristic that at $\sigma\to 0$ (34) automatically transforms to a conventional

$$k_r = \omega\sqrt{\frac{\rho}{T_l}} . \tag{35}$$

In resistant lines, the wave delay τ_r depends on frequency ω in a complex way. The same as in case with χ, the resistance effect on k_r is especially sensible at low frequencies, when σ/ω was comparable with ρ. At high frequencies, (34) can be approximately presented as

$$k_r \simeq \omega\sqrt{\frac{\rho}{T_l}}\left(1+\frac{\sigma^2}{8\rho^2\omega^2}\right). \tag{36}$$

We can see from (34) and (36) that k_r also depends on $\sigma/\rho\omega$.

To obtain the sought solution for a distributed line, we have to determine additionally the limit value φ_{0r} characterising the wave delay in a line as a whole. Substituting the values A and B from (30) to (18), we obtain

$$\varphi_{0r} = \arcsin\sqrt{\frac{\sqrt{\omega^2\rho^2+\sigma^2}-\omega^2\rho^2}{2\sqrt{\omega^2\rho^2+\sigma^2}}} . \tag{37}$$

As we see from (37), the same as in previous cases, the line resistance influence on φ_{0r} is limited by the low-frequencies band and depends on $\sigma/\rho\omega$ too. In this connection, we can think this ratio determining the effect of resistance on the pattern of vibration process in an elastic distributed line.

The typical plot $\varphi_{0r}(\omega)$ is shown in Fig. 5. It fully corroborates the above analysis of φ_{0r}. We can add only one more feature. To determine the resulting phase delay of vibration process as a whole, we have to consider φ_{0r} in (15) together with the complex unity $(-j)$, which displaces the solution by $(-\pi/2)$. Due to it, for a resistant line at $\omega\to 0$ the total phase delay tends to $(-\pi/4)$, while in an ideal line it is $(-\pi/2)$. With growing frequency, at small resistance, the total phase delay equalises with the value typical for an ideal line. However at large resistances this process naturally retards.

On the basis of determined parameters R, φ_{0r} and k_r we can write the general solution describing the vibration process in a semi-infinite distributed elastic line in presence of resistance σ. Substituting (32), (34) and (37) to (15) and using (29), we yield

$$x = x_0 - \frac{F_0 e^{j\varphi_{0r}} e^{\chi x_0}}{\sqrt{T_l}\left(\rho^2\omega^4+\sigma^2\omega^2\right)^{1/4}} e^{j(\omega t+\varphi_0-k_r x_0)}. \tag{38}$$

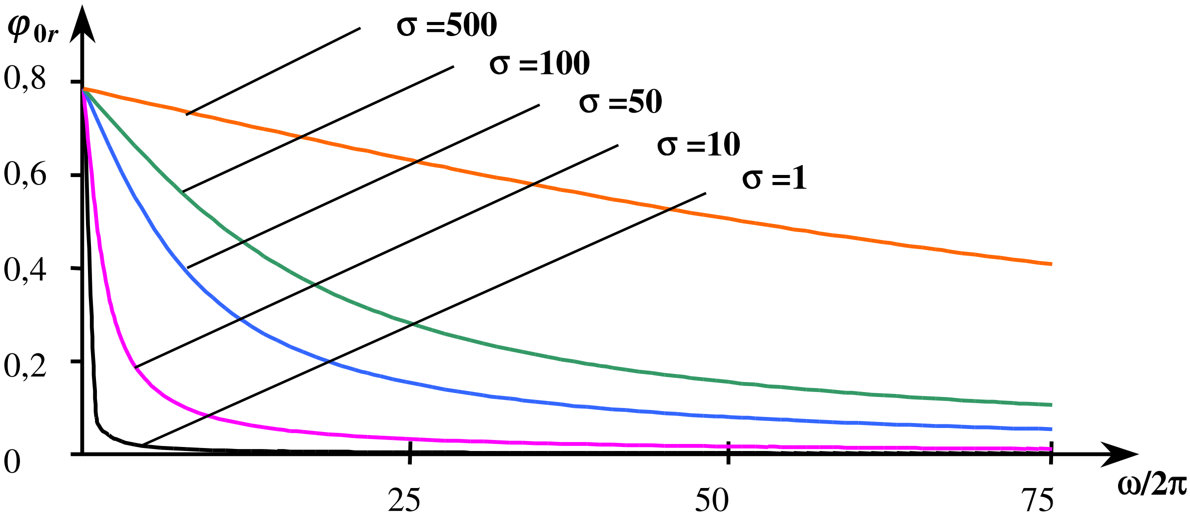

Fig. 5. The general phase φ_{0r} displacement of vibrations in a distributed line, with respect to the external action frequency ω, at different values of specific resistance σ

The expression (38) has retained the main features of (15). The same as in (15), the resistance effects both on the vibration in a line as a whole and on the along-the-line excitation transfer, and this effect is especially strong at low and ultralow frequencies. This connection of solutions is quite natural, since the solutions for a distributed line correspond to an initial frequency band of solutions for a lumped line. As shown in [13], to model a lumped line by means of that distributed is permissible at a condition

$$\frac{a}{\lambda} << \frac{1}{\pi} . \tag{39}$$

If disregarding this condition, the solutions (38) will lose their accuracy of the line process description. Particularly, (38) does not describe the processes at the bands of boundary and overcritical frequencies.

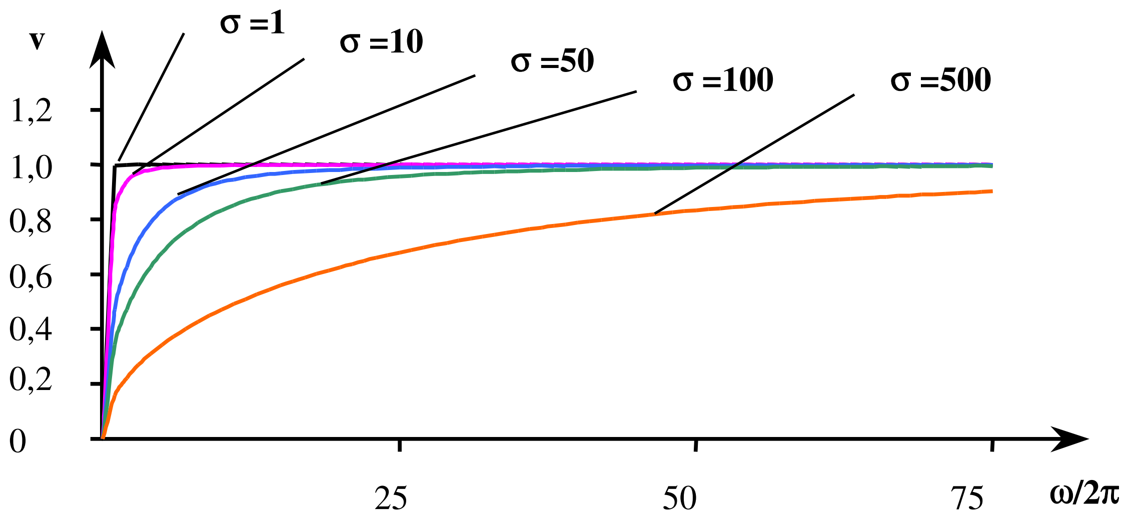

Fig. 6. The phase velocity v of wave propagation in a distributed line, with respect to the external action frequency ω, at different values of specific resistance σ

We can trace the same feature in the phase velocity dependence on frequency. To find it, substitute (34) into (27). We obtain

$$v_f = \sqrt{\frac{T_l}{\rho}} \frac{\sqrt{2}}{\sqrt{1+\sqrt{1+\sigma^2/\omega^2\rho^2}}} . \tag{40}$$

At $\omega \to 0$, the phase velocity $v_{ph} \to 0$, and at $\sigma^2/\omega^2\rho^2 << 1$, $v_{ph} \to \sqrt{T_l/\rho}$ – i.e., to the phase velocity conventional value in an ideal distributed line. While in an ideal distributed line the phase velocity does not vanish at $\omega \to 0$, but keeps the value $\sqrt{T_l/\rho}$. In its turn, (40) does not describe the anomalous dispersion of velocity at high frequencies. The typical plot of $v_f(\omega)$ is presented in Fig. 6.

Thus we see that in (38) the main regularities of the resistance effect on vibration process have retained. At the same time, the regularities describing the main parameters φ_r, φ_{0r}, R, v_{ph} have essentially transformed. In them, as a result of limit passing, the summands describing the process at the boundary band have disappeared, and this transformation is irreversible in case of the reverse transition to a lumped line. This feature makes impossible the correct description of a lumped line by means of that distributed.

5. Comparison with conventional theoretical and experimental results

The problem of vibrations in a homogeneous chain of elastically linked bodies has a broad application to ultrasonic vibrations in gas, liquid and solid. One of multiple fine surveys of theoretical and experimental results in this area is given in [16]. Ibidem [16, p.45] a typical plot of the phase velocity c with respect to pressure p and temperature T is given for the critical domain of the gas-liquid-solid transition in the carbonic acid gas (see Fig. 7). It has been plotted on the basis of acoustic waves experimentally studied at the ultrasonic band. This plot shows the anomalous dispersion of velocity in a low-pressure domain; an abrupt minimum of velocity whose location depends on temperature, and a fair growth of the velocity at a high-pressure domain. Many investigations were devoted to explain such complex regularity; the theories by Altenburg [17], Kudryavtcev [18] and Fisher [19] stand out among them.

Altenburg in his model considers an 1D chain of equidistant molecules under condition that each of them interacts only with two neighbouring molecules. Naturally he comes to the conventional solution like

$$u_n = ue^{j(\omega t - n\delta)} \tag{41}$$

(here $\delta = 2\arcsin \omega/\omega_0$ and $\omega_0 = 2\sqrt{s/m}$). Further, reducing this solution to a nonlinear description of the above regularity of velocity with respect to pressure and temperature, Altenburg considers the sound propagating in a face-centred lattice, though he studies a liquid model.

Kudryavtcev [18] proceeds from the following relation between adiabatic compressibility of substance and its internal energy U:

$$K = \frac{1}{\beta} = -V\left(\frac{\partial p}{\partial V}\right)_S = V\left(\frac{\partial^2 U}{\partial u^2}\right)_S , \tag{42}$$

where S is the entropy, and β in this case is the compressibility of substance.

Kudryavtcev also supposes a full order of molecules and calculates the derivative of (42) with the variation of intermolecular distance along some direction. In fact, he also reduces the problem to the vibration of 1D chain of elastically linked molecules. Further, supposing that the interior energy of liquid is the potential energy of the intermolecular action and that when calculating, one can consider only the neighbouring molecules interaction, he obtains

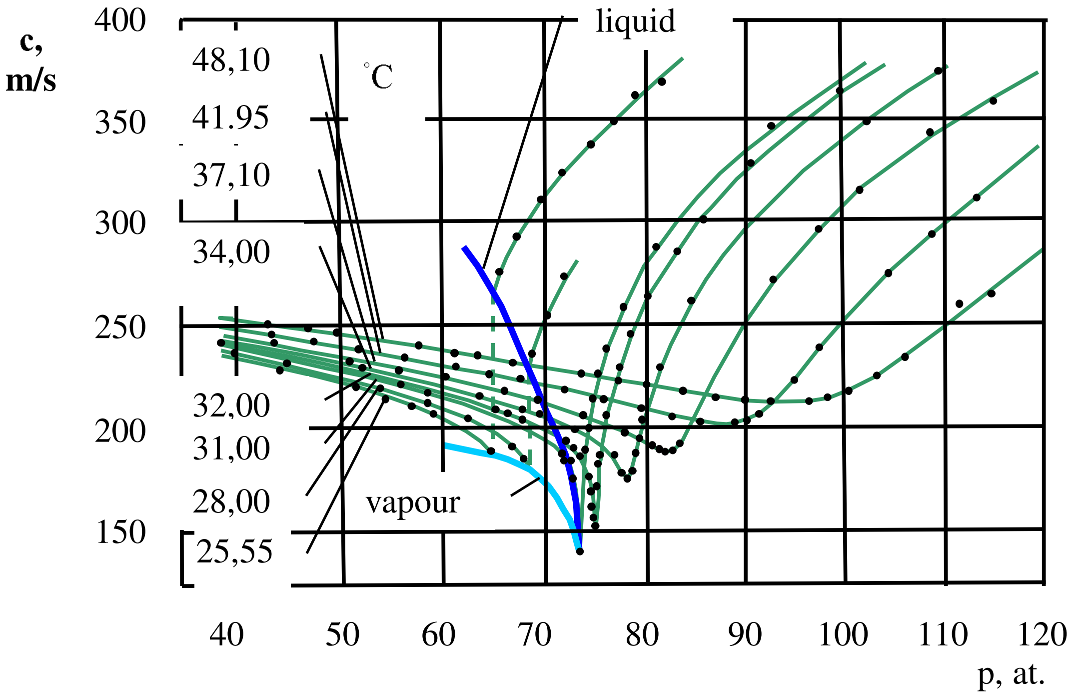

Fig. 7. The sound velocity in the critical domain as the function of pressure [16, p.45]

\- - - - - - - - the sound velocity along the saturation line in liquid and gas

$$\frac{1}{\beta}=\frac{\varphi''(r)}{r}, \tag{43}$$

where the relationship of the interaction potential $\varphi(r)$ is taken after Lennard – Jones. "Selecting the equations numerical coefficients, one can achieve in this case a satisfying conformity of the measured and calculated values of the sound velocity" [16, pp.89-90].

Finally, "Fisher has rigorously solved the sound propagation problem in an 1D chain of interacting particles, considering a disordered chain of particles located along the axis x " [16, p.90]. Fisher has found his model on the state equation, basing it on the potential of forces acting between a pair of particles

$$l=-\,kT\frac{\partial\psi(x)}{\partial t}, \tag{44}$$

where l is the average distance between a pair of the nearest particles, and enthalpy equation

$$\frac{\hat{H}}{\hat{N}}=\frac{1}{2}kT+k\frac{\partial}{\partial T}\left(T^2\frac{\partial\psi}{\partial t}\right). \tag{45}$$

Here k is Boltzman constant, and

$$\psi=\ln\int_0^\infty e^{-(px+\varphi(x))/kT}dx. \tag{46}$$

With it he has obtained the expression for the sound velocity squared as

$$\psi=\ln\int_0^\infty e^{-(px+\varphi(x))/kT}dx. \tag{47}$$

“In the most general sense the expressions (46) and (47) solve perfectly exact the problem of the sound velocity calculation in an 1D liquid as a function of temperature and pressure at the given potential $\varphi(x)$. As we see, already in 1D case the expression for sound velocity is very complex”

[16, p.91]. So it can be rigorously solved only in two extreme cases $T \to 0$ and $T \to \infty$. In the first case one obtains the results comparable with the solutions by Altenburg and Kudryavtcev.

At the same time, no one of presented theories takes into account the following factors: the nonlinear influence of resistance on the vibration process parameters and the features of vibration processes in an overcritical band. However, comparing the experimental data in Fig. 7 with the results presented in Fig. 3, one can easily see that the characteristics are comparable.

First of all, we see in Fig. 7 that in the low-pressure domain the anomalous dispersion of velocity takes place, increasing as approaching to the critical regime (the point of minimal value of the velocity), the same as in Fig. 3. The curves inclination after the extreme point is approximately equal too.

Second, taking a chain of the elastically connected molecules as the basic model, we can suppose that the increasing stiffness of links promotes the growing pressure at constant temperature. With it, one and the same external excitation frequency corresponds to a more high-frequency band in Fig. 3, due to the diminishing boundary frequency ω_0. Thus, we can present the plot $c(p)$ in Fig. 7 as a plot $c(\omega)$.

Third, it is known that the medium temperature increasing at constant pressure can be achieved only at the diminishing density of medium. It leads to the increase of intermolecular distance a and diminishing of the link stiffness s. In its turn, according to (15), it leads to the increase of the boundary frequency and phase velocity in accord with (27). Thus, the temperature growth will displace the boundary frequency ω_0 to the high-frequency band.

And the last, with the growing temperature, the elastic system resistance will naturally increase.

Summing up the above key points, we can explain qualitatively the curves behaviour in Fig. 7, using the regularities of the plot in Fig. 3.

Each of dependencies $c(p)$ in Fig. 7 has its distinctive minimum at the boundary frequency ω_0. When the temperature grows, this minimum displaces to the increasing pressure (analogous to the higher frequency), and the sharpness of maximum is smoothed, because of increasing resistance of a line. With it the wave propagation velocity increases, because of increasing intermolecular distance a. The value of this distance has not an effect on the solution (15), but it effects directly on the phase velocity. But the link stiffness effects both on (15) and on the phase velocity through φ_r. These factors effect oppositely on the velocity. This is why we can expect the velocity to be negligibly growing with temperature. It will depend much both on the elastic system parameters and on the relationship of the influences between the intermolecular distance growing and link stiffness diminishing.

From the presented brief qualitative analysis we can see that if taking into consideration the resistance influence on the line vibration processes, we can essentially refine the conventional models and promote them to correspond better the experimental data.

6. Conclusions

Studying the vibrations in a semi-infinite resistant elastic lumped and distributed lines, we have revealed that the resistance effects both on the line vibration pattern as a whole and on the parameters of the along-the-line excitation transmission. The resistance influence on vibrations as a whole reveals at the low and ultralow frequencies, and its effect on the excitation transmission is essential at all the range.

We have ascertained that the progressive vibrations in a resistant line retain also at the overboundary frequencies. At this band the phase velocity grows approximately linearly, and the group velocity increases when the resistance diminishing. At large r, the dependence of all parameters of vibration process on frequency transforms into the smooth functions without local extremums or kinks in the domain of critical vibration regime corresponding to the boundary frequency ω_0.

Furthermore, we have revealed that when the vibration propagates in a line, the phase delay never reaches the value π corresponding to the first Brillouin zone.

The solutions for a distributed line reflect the salient features of the initial frequency band of solutions for a lumped line. Particularly, the phase velocity vanishes at $\omega \to 0$, not retaining its finite value, as in an ideal elastic line. At the same time, the solutions for a distributed line do not describe the anomalous dispersion of velocity, which the solutions for a resistant line describe.

When comparing the obtained results with the experimental data investigating the ultrasonic wave phase velocity in the carbonic acid gas, we see that taking the resistance into account, we can essentially refine the conventional models and promote their better correspondence to the experimental data.

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