WHEN SIZE MATTERS: SUBSHIFTS AND THEIR RELATED TILING SPACES

ALEX CLARK AND LORENZO SADUN

ABSTRACT. We investigate the dynamics of substitution subshifts and their associated tiling spaces. For a given subshift, the associated tiling spaces are all homeomorphic, but their dynamical properties may differ. We give criteria for such a tiling space to be weakly mixing, and for the dynamics of two such spaces to be topologically conjugate.

1. INTRODUCTION

We consider the dynamics of 1-dimensional minimal substitutions subshifts (with a natural $\mathbb{Z}$ action) and the associated 1-dimensional tiling spaces (with natural $\mathbb{R}$ actions). Given an alphabet $\mathcal{A}$ of n symbols $\{a_1, ..., a_n\}$, a *substitution* on $\mathcal{A}$ is a function σ from $\mathcal{A}$ into the non-empty, finite words of $\mathcal{A}$. Associated with such a substitution is the $n \times n$ matrix M which has as its (i, j) entry the number of occurrences of a_j in $\sigma(a_i)$. A substitution is *primitive* if some positive power of M has strictly positive entries. The substitution σ induces the map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ given by $\langle \ldots u_{-1}.u_0 u_1 \ldots \rangle \overset{\sigma}{\mapsto} \langle \ldots \sigma(u_{-1}) . \sigma(u_0) \sigma(u_1) \ldots \rangle$. For any primitive substitution σ there is at least one point $u \in \mathcal{A}^{\mathbb{Z}}$ which is periodic under σ, and the closure of the orbit of any such u under the left shift map s of $\mathcal{A}^{\mathbb{Z}}$ forms a minimal subshift $\mathcal{S}$. This subshift $\mathcal{S}$ is uniquely determined by σ, and the restriction $\sigma|_{\mathcal{S}}$ is a homeomorphism onto $\mathcal{S}$. To avoid trivialities, we shall only consider *aperiodic* substitutions σ, those for which the subshift $\mathcal{S}$ is not periodic.

Given a collection of intervals $\mathcal{I} = \{I_1, ..., I_n\}$, a *tiling* T of $\mathbb{R}$ by $\mathcal{I}$ is a collection of closed intervals $\{T_i\}_{i\in\mathbb{Z}}$ satisfying

1. $\cup_{i\in\mathbb{Z}} T_i = \mathbb{R}$,
2. For each $i \in \mathbb{Z}$, T_i is the translate of some $I_{\tau(i)} \in \mathcal{I}$, and
3. $T_i \cap T_{i+1}$ is a singleton for each $i \in \mathbb{Z}$.

If the function $\tau : \mathbb{Z} \to \{1, ..., n\}$ is an element of a minimal substitution subshift $\mathcal{S}$ of $\{1, ..., n\}^{\mathbb{Z}}$, then T is called a *substitution tiling.* There is a natural topology on the space $\mathfrak{T}$ of tilings of $\mathbb{R}$ that is induced by a metric which measures as close any two tilings T and T' that agree on a large neighborhood of 0 up to an ε translation. There is then the continuous translation action $\mathbf{T}$ of $\mathbb{R}$ on $\mathfrak{T}$: for $t \in \mathbb{R}$ and the tiling $T = \{T_i\}_{i\in\mathbb{Z}}$, $\mathbf{T} : (T, t) \mapsto \mathbf{T}_t T = \{T_i - t\}_{i\in\mathbb{Z}}$. (A positive element of $\mathbb{R}$ moves the origin to the right, or equivalently moves tiles to the left). The closure of the translation orbit of any substitution tiling T in $\mathfrak{T}$ is then a minimal set of the action, the (*substitution*) *tiling space* $\mathcal{T}$ of the tiling T. As the tiling space for any iterate of a substitution σ is the same as the tiling space of σ and since any substitution has a point u whose right half

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$\langle u_0u_1\dots\rangle$ is periodic under substitution, for ease of discussion we shall only consider substitutions with a point u whose right half $\langle u_0u_1\dots\rangle$ is fixed under substitution.

Flows under a function provide an alternative description of tiling spaces convenient for our purposes. Given a minimal subshift $(\mathcal{S}, s)$ of the shift on $\mathcal{A}^{\mathbb{Z}}$ and $f : \mathcal{A} \to (0,\infty)$, there is the flow under f given by the natural $\mathbb{R}$ action on $\mathcal{T}_f = \mathcal{S}\times\mathbb{R}/\sim$, where $(u, f(u_0)) \sim (s(u), 0)$. If we then associate with each $a_i \in \mathcal{A}$ a closed interval I_i of length $f(a_i)$ and form the tiling on T of $\mathbb{R}$ by $\{I_1,\dots,I_n\}$ with associated function $\tau = u \in \mathcal{A}^{\mathbb{Z}}$, which has the left endpoint of the interval corresponding to u_0 at $0 \in \mathbb{R}$, then the function sending T to the class of $(u,0)$ in $\mathcal{T}_f$ extends uniquely to a homeomorphism $\mathcal{T} \to \mathcal{T}_f$ which conjugates the respective $\mathbb{R}$ actions.

The primary focus of this paper is the extent to which the dynamical systems $\mathcal{T}_f$ depend on the function f. If all we care about is the topological space, they don't:

Theorem 1.1. *Let $\mathcal{S}$ be a subshift, and let f, g be two positive functions on the alphabet of $\mathcal{S}$. Then $\mathcal{T}_f$ and $\mathcal{T}_g$ are homeomorphic.*

Proof. Every class $x \in \mathcal{T}_f$ has a (unique) representative of the form (u,t), with $0 \le t < f(u_0)$. Let $h(x)$ be the class of $(u, g(u_0)t/f(u_0))$ in $\mathcal{T}_g$. As the function on quotient spaces induced by a continuous function $\mathcal{S}\times\mathbb{R} \to \mathcal{S}\times\mathbb{R}$, h is continuous, and it has a well-defined continuous inverse. □

But what about dynamics? Dekking and Keane [DK] showed that, for the substitution $a \to abab$, $b \to bbba$, the associated tiling space with $f(a) = 2, f(b) = 1$ has point spectrum only at multiples of 2π . (That is, the associated discrete dynamical system is weakly mixing.) Further, Berend and Radin [BR] showed that a tiling space based on this substitution has trivial point spectrum if $f(a)/f(b)$ is irrational. Radin and Sadun [RS] obtained very different results for the Fibonacci substitution $a \to b$, $b \to ab$. For the Fibonacci substitution, $\mathcal{T}_f$ is topologically conjugate to $\mathcal{T}_g$ whenever $f(a) + \tau f(b) = g(a) + \tau g(b)$, where $\tau = (1+\sqrt{5})/2$ is the golden mean. (The conjugacy is not the map h, described above, but is homotopic to h. A generalization of this construction appears below in Section 3.) In particular, all Fibonacci tiling spaces have pure point spectrum, regardless of the sizes of the tiles.

In Section 2 we examine the general ergodic properties of the flows $\mathcal{T}_f$, showing that these flows are never mixing. We also provide general criteria for the existence of point spectrum and constraints on the form of that spectrum, in terms of the eigenvalues of the substitution matrix and the ratios of lengths of the tiles. The Dekking-Keane and Berend-Radin results are then easily understood. In Section 3 we provide criteria for $\mathcal{T}_f$ and $\mathcal{T}_g$ to be topologically conjugate.

Sadun and Williams [SW] have shown that every finite dimensional tiling space meeting mild conditions (consisting of a finite number of polygonal tiles, appearing in a finite number of orientations, and meeting edge-to-edge) can be deformed into a tiling space that is topologically conjugate to the d-fold suspension of a $\mathbb{Z}^d$ subshift. This deformation changes the shapes and sizes of the tiles, but not the combinatorics of which tiles touch which others. The difference between the dynamics of general tiling spaces and of subshifts is therefore not combinatorial in nature, but depends rather on the geometry of the individual tiles. This paper is an exploration of this phenomenon in one dimension, where the geometry of a tile is simply its length. Moreover, our results have implications for the large class of higher dimensional substitution tiling spaces that can be realized by taking the product of one-dimensional substitution tiling spaces.

2. Ergodic Properties

For a substitution tiling space $\mathcal{T}_f$, let $L = (f(a_1), \ldots, f(a_n))$ be the (row) vector that gives the lengths of the various tiles. Let L_0 be the normalized positive left Perron-Frobenius eigenvector with eigenvalue λ_{PF} of the substitution matrix M, and let $\mathcal{T}_0$ be the tiling space associated with the length vector L_0. Every such tiling space $\mathcal{T}_0$ is known to admit a self-homeomorphism derived from σ (also denoted σ) which inflates all the tiles by a factor of λ_{PF} and which maps the class of a point $(x, 0)$ to the class of $(\sigma(x), 0)$, see, e.g., [BD]. The more general substitution tiling space $\mathcal{T}_f$ then admits a substitution homeomorphism, also denoted σ, conjugate to the substitution homeomorphism of $\mathcal{T}_0$ via the homeomorphism h of theorem 1.1, but this substitution homeomorphism is not generally affine. Solomyak [S] has investigated the ergodic properties of the self-affine $\mathcal{T}_0$ in the case that the matrix M is diagonalizable. Now we proceed to investigate the ergodic properties of the more general tiling spaces $\mathcal{T}_f$, requiring only that the substitution be primitive and aperiodic. As most proofs in this section are similar to those for analogous known results for substitutions, the proofs shall be abbreviated.

It should be noted that most of the literature is devoted to the ergodic properties of one-sided substitutions, while we are dealing with two-sided substitutions. However, as there are at most n^2 $s-$orbits of words in a one-sided substitution that admit more than one left extension [BDH], measure theoretic considerations are not altered. For example, by placing all two-sided words with the same right halves into the same partition element, partitions of one-sided substitutions lead to partitions of two-sided substitutions with corresponding partition elements having the same measure.

Definitions. For a word $w = w_0 \cdots w_k$ from the alphabet $\mathcal{A}$ of the substitution subshift $\mathcal{S}$ derived from the substitution σ,

$$[w] \stackrel{\text{def}}{=} \{u \in \mathcal{S} \mid u_0 \cdots u_k = w\}. \tag{1}$$

A *cylinder set* $[w] \times I$ in $\mathcal{T}_f$ for a word $w = w_0 \cdots w_k$ and interval $I \subseteq [0, f(w_0))$ is the set of all $x \sim (u, t) \in \mathcal{T}_f$ with $u \in [w]$ and $t \in I$.

The *population vector* $v = (v_1, ..., v_n)^T$ of the finite word w gives the number of occurrences v_i of the letter a_i in w. Note that v is a column vector, not a row, and that the substitution matrix acts from the left.

For $u \in \mathcal{A}^{\mathbb{Z}}$, a *recurrence word* is a finite word w in u, $w = u_r u_{r+1} \cdots u_s$ satisfying the condition that $u_{s+1} = u_r$, and a *recurrence vector* is the population vector for a recurrence word. For example, for $\ldots abbcba \ldots$ in $\{a, b, c\}^{\mathbb{Z}}$, $abbcb$ is a recurrence word for which $(1, 3, 1)^T$ is the recurrence vector. The vectors $(0, 1, 0)^T$, $(0, 1, 1)^T$, and $(0, 2, 1)^T$ are recurrence vectors for the successive b's.

A recurrence vector v is called *full* if the vectors $\{M^k v\}$, with k ranging from 0 to $n-1$, are linearly independent. If M is diagonalizable, this is equivalent to v pairing nontrivially with every left-eigenvector of M. Note that every substitution on two letters admits a full recurrence vector, namely $(0,1)^T$ or $(1,0)^T$.

Primitive aperiodic substitution tilings are recurrent, meaning that any finite patch of one tiling appears somewhere in every other tiling. As a result, the set of recurrence vectors is the same for every tiling in the space, and so we can speak of the recurrence vectors of a tiling *space*.

The following lemma and its implication for mixing are proved in much the same way as for substitutions [DK, Theorem 2].

Lemma 2.1. *Let v be the recurrence vector of a recurrence word r and let $t_m \stackrel{def}{=} LM^m v$. For any cylinder set $[w] \times I$ in $\mathcal{T}_f$ and for any m, let $S_m \stackrel{def}{=} ([w] \times I) \cap (\mathbf{T}_{-t_m}([w] \times I))$. Then*

$$\liminf_{m\to\infty} \mu(S_m) \geq \gamma \cdot \mu([w] \times I), \tag{2}$$

where γ is a positive constant independent of w and I, depending only on the recurrence word r of v.

Proof. With $t'_m \stackrel{\text{def}}{=} (1,\ldots,1) M^m v$, it follows from [DK] that the measures μ' in the substitution subshift of the sets $S'_m \stackrel{\text{def}}{=} [w] \cap s^{-t'_m}[w]$ satisfy

$$\liminf_{m\to\infty} \mu'(S'_m) \geq \gamma' \cdot \mu'([w]), \tag{3}$$

for some positive constant γ' independent of w. Observe that $\mu'([w])$ is the limit as $k \to \infty$ of the average number of iterates $\{s^i(u) \,|\, i = 0, ..., k-1\}$ in $[w]$, for any given $u \in \mathcal{A}^{\mathbb{Z}}$. And the measure $\mu([w] \times I)$ in $\mathcal{T}_f$ is the limit as $T \to \infty$ of the average amount of time that $\{\mathbf{T}^t(x) \,|\, t \in [0,T]\}$ spends in $\mu([w] \times I)$ for any given $x \in \mathcal{T}_f$. Hence, $\mu([w] \times I) = \mu'([w]) \times \dfrac{\text{length}(I)}{\mu'([a_1]) f(a_1) + \cdots + \mu'([a_n]) f(a_n)}$. Now, if $u \in [w]$ and $t \in I$, then $s^{t'_m}(u) \in [w]$ if and only if $\mathbf{T}_{t_m}((u,t)) \in [w] \times \{t\}$,and the result follows. □

By choosing the cylinder set smaller in measure than a γ for a fixed recurrence word, this leads to the following conclusion.

Theorem 2.2. *None of the substitution tiling space flows on $\mathcal{T}_f$ are (strongly) mixing.*

Theorem 2.3. *The number k is in the point spectrum of $\mathcal{T}_f$ if and only if for any recurrence vector v,*

$$\frac{k}{2\pi} LM^m v \to 0 (\textit{mod } 1) \textit{ as } m \to \infty. \tag{4}$$

Moreover, when this condition is met there is a constant C and a $\rho < 1$ satisfying

$$|\exp(ikLM^m v) - 1| < C\rho^m \tag{5}$$

for any given v.

Proof.
Essentially the same arguments as found in [FMN] using Rokhlin stacks and columns apply to show that the condition is both necessary and sufficient for *return words,* finite words w in u, $w = u_r u_{r+1} \cdots u_s$ satisfying for sufficiently large m the conditions $\sigma^m u_r = \sigma^m u_{s+1}$ and $\sigma^m u_r \neq \sigma^m u_j$ for $j = r+1, ..., s$. Since the eigenfunction constructed as in [FMN] using the geometric convergence and the condition for return words is continuous, one need only consider continuous eigenfunctions.

We now show that for continuous eigenfunctions the condition for the general *recurrence* word is necessary, which is all that need be shown since the condition with return words is already sufficient. Consider then a continuous eigenfunction f with eigenvalue k. Let $x \sim (u, 0) \in \mathcal{T}_f$, where $u \in \mathcal{A}^{\mathbb{Z}}$ is a point of the substitution σ with fixed right half $\langle u_0 u_1 \ldots \rangle$, and let v be a recurrence vector for a recurrence word between u_0 and some u_s, and let $t_m \stackrel{\text{def}}{=} LM^m v$. Note that $\mathbf{T}_{t_0} x$ agrees with x on the interval covered by the single tile of x represented by u_0. Then $\mathbf{T}_{t_m} x = \sigma^m(\mathbf{T}_{t_0} x)$ agrees with x on forward time intervals of increasing unbounded length, and so $\exp(ikt_m)$ must converge to 1, and $kt_m/(2\pi)$ must converge to 0 (mod 1). Now let v_1 and v_2 be two such recurrence vectors. Since $kLM^m v_1/(2\pi)$ and $kLM^m v_2/(2\pi)$ both converge to zero (mod 1), so does $kLM^m(v_1 - v_2)/(2\pi)$. In other words, we can use any recurrence vector for the type of letter (say, a) that occurs at u_0. Finally, suppose that v_3 is a recurrence vector for letter b. Since σ is primitive, suppose a appears somewhere in the ℓ-th substitution of b, for some integer ℓ. Then $v_4 = M^\ell v_3$ is a recurrence vector for a, so $kLM^m v_3/(2\pi) = kLM^{m-\ell} v_4/(2\pi)$ converges to $0 (\text{mod } 1)$. To obtain the geometric convergence as in the statement of the theorem, one applies the argument of [H, Lemme 1] with the lattice $f(a_1)\mathbb{Z} \oplus \cdots \oplus f(a_n)\mathbb{Z}$ in place of the lattice $\mathbb{Z}^n$. □

The application of this criterion depends on the eigenvalues of M and on the possible forms of the recurrence vectors.

Theorem 2.4. *Suppose that all the eigenvalues of M are of magnitude 1 or greater, and that there exists a full recurrence vector. If the ratio of any two tile lengths is irrational, then there is trivial point spectrum. If the ratios of tile lengths are all rational, then the point spectrum is contained in $2\pi\mathbb{Q}/L_1$.*

Proof. Let k be in the point spectrum, and consider the sequence of real numbers $t_m = kLM^m v/(2\pi)$, where v is a fixed full recurrence vector. Let $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ be the characteristic polynomial of M. Note that the a_i's are all integers, since M is an integer matrix. Since $p(M) = 0$, the t_m's satisfy a recurrence relation:

$$t_{m+n} = -\sum_{k=0}^{n-1} a_k t_{m+k}. \tag{6}$$

By Theorem 2.3, the t_m's converge to zero mod 1. That is, we can write

$$t_m = i_m + \epsilon_m \tag{7}$$

where the i_m's are integers, and the ϵ_m's converge to zero as real numbers. By substituting the division (7) into the recursion (6), we see that both the i's and the ϵ's must separately satisfy the recursion (6), once m is sufficiently large (e.g., large enough that the ϵ's are bounded by $1/\sum|a_i|$). However, any solution to this recursion relation is a linear combination of powers of the eigenvalues of M (or polynomials in m times eigenvalues to the m-th power, if M is not diagonalizable). Since the eigenvalues are all of magnitude one or greater, such a linear combination converges to zero only if it is identically zero. Therefore ϵ_m must be identically zero for all sufficiently large values of m.

Each t_m is an integer linear combination of the elements of the vector $kL/(2\pi)$. From a sequence of n consecutive s's, one can recover all the elements of kL by inverting an integer matrix whose columns are powers of M times v. (The invertibility of this matrix depends on the fact that v is full and that zero is not an eigenvalue.) By Cramer's rule, this inversion involves integer multiplication and addition, followed by division by the determinant of this matrix. Since the t_m's are integers (for m large enough), the components of $kL/(2\pi)$ must then all be rational. If L_i/L_j is irrational, this implies that $k=0$. If all the L's are rationally related, it implies that k lies in $2\pi\mathbb{Q}/L_1$. □

Theorem 2.4 gives partial spectral information. When the alphabet has only two letters and the substitution has constant length, we can say considerably more:

Theorem 2.5. *Suppose that we have a primitive substitution on two letters of constant length n, where $\sigma(a)$ contains n_a a's and $n-n_a$ b's, while $\sigma(b)$ contains n_b a's and $n-n_b$ b's. Suppose further that $1\le n_a,n_b\le n-1$ and $n_a\ne n_b$. Let z be the greatest common factor of n and n_a-n_b. Then the point spectrum σ_{pp} depends as follows on the ratio L_1/L_2:*

1. *If $L_1=L_2$, then there is a positive integer N such that*

$$N\mathbb{Z}[1/n]\subset NL_1\sigma_{pp}/2\pi\subset\mathbb{Z}[1/n]. \tag{8}$$

2. *If $L_1/L_2\in\mathbb{Q}-\{1\}$, then there exist positive integers N_1 and N_2 such that*

$$N_1\mathbb{Z}[1/z]\subset N_2L_1\sigma_{pp}/2\pi\subset\mathbb{Z}[1/z]. \tag{9}$$

3. *If $L_1/L_2\notin\mathbb{Q}$, then $\sigma_{pp}=\{0\}$.*

Proof. The $L_1=L_2$ case is not new – Coven and Keane [CK] proved an even stronger result over 30 years ago. We prove all three cases together, since the proof neatly illustrates the significance of the ratio L_1/L_2.

The substitution matrix

$$M=\begin{pmatrix} n_a & n_b \\ n-n_a & n-n_b\end{pmatrix} \tag{10}$$

has eigenvalues n and n_a-n_b, with left-eigenvectors $(1,1)$ and $(n-n_a,-n_b)$ and right-eigenvectors $(n_b,n-n_a)^T$ and $(1,-1)^T$. Since $n_a\ne n_b$, both eigenvalues have magnitude one or greater.

Since the system is aperiodic and there are only two letters, either $(1,0)^T$ or $(0,1)^T$ is a (full) recurrence vector. Suppose that $(1,0)^T$ is (the other case is similar). Then we take

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{n+n_b-n_a}\begin{pmatrix} n_b \\ n-n_a \end{pmatrix} + \frac{n-n_a}{n+n_b-n_a}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{11}$$

As in the proof of Theorem 2.4, if $k \in \sigma_{pp}$ then the numbers $t_m = kLM^m v/2\pi$ must eventually be integers. Note that

$$\begin{aligned} 2\pi t_m =& kLM^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 2\pi t_{m+1} =& kLM^{m+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = kLM^m \begin{pmatrix} n_a \\ n-n_a \end{pmatrix}, \end{aligned} \tag{12}$$

so

$$2\pi(t_m, t_{m+1}) = kLM^m \begin{pmatrix} 1 & n_a \\ 0 & n-n_a \end{pmatrix}, \tag{13}$$

and

$$kLM^m = 2\pi(t_m, t_{m+1}) \begin{pmatrix} 1 & n_a \\ 0 & n-n_a \end{pmatrix}^{-1} = \frac{1}{n-n_a}\left((n-n_a)t_m, t_{m+1} - n_a t_m\right). \tag{14}$$

Multiplying on the right by the right-eigenvectors of M gives two scalar equations for k:

$$\begin{aligned} k(L_1 - L_2)(n_a - n_b)^m =& \frac{2\pi}{n-n_a}(nt_m - t_{m+1}) \\ k(n_b L_1 + (n-n_a)L_2)n^m =& 2\pi((n_b - n_a)t_m + t_{m+1}). \end{aligned} \tag{15}$$

If $L_1 = L_2$, the first equation reads $0 = 0$, but the second equation says that $NkL_1 n^m/2\pi$ is an integer for m sufficiently large, where $N = n + n_b - n_a$. Thus $NL_1\sigma_{pp}/2\pi \subset \mathbb{Z}[1/n]$.

If L_1/L_2 is rational but not one, then the first equation says that $\sigma_{pp}L_1/2\pi$ is contained in a fixed rational multiple of $\mathbb{Z}[1/(n_a - n_b)]$, while the second says that $\sigma_{pp}L_1/2\pi$ is contained in a fixed rational multiple of $\mathbb{Z}[1/n]$. Thus an integer multiple of $\sigma_{pp}L_1/2\pi$ lies in $\mathbb{Z}[1/n] \cap \mathbb{Z}[1/(n_a - n_b)] = \mathbb{Z}[1/z]$.

If L_1/L_2 is irrational, then the only simultaneous solution to both equations is $k = 0$. For if $k \neq 0$ is a solution, then the ratio of the two equations would imply that $(L_1 - L_2)/(n_b L_1 + (n-n_a)L_2)$ would be rational, which is inconsistent with the irrationality of L_1/L_2. Thus $\sigma_{pp} \subset \{0\}$ and so $\sigma_{pp} = \{0\}$. All that remains is to show that $L_1\sigma_{pp}/2\pi$ contains $\mathbb{Z}[1/n]$ if $L_1 = L_2$, and that $L_1\sigma_{pp}/2\pi$ contains a rational multiple of $\mathbb{Z}[1/z]$ if L_1/L_2 is rational but not one.

We call the result of applying the substitution ℓ times to a single tile (or a single letter) a *supertile* of order ℓ. If $L_1 = L_2$, then all supertiles of order m have length $n^m L_1$. Thus the coordinates of the endpoints of such supertiles agree modulo $n^m L_1$. For each pair of integers (j, m) we define a function on our tiling space. If x is a tiling, let

$$\psi_{j,m}(x) = \exp(2\pi i jp/n^m L_1), \tag{16}$$

where p is the endpoint of *any* supertile of order m in x. This is manifestly an eigenfunction of translation with eigenvalue $2\pi j/n^m L_1$.

If L_1/L_2 is rational but not integral, then the length of each supertile of order m is a fixed rational linear combination of $L_1 n^m$ and $L_1(n_a - n_b)^m$. Thus there is a constant c, a rational multiple of L_1, such that every supertile of order m has length divisible by cz^m. For each pair (j,m) we then define the eigenfunction

$$\phi_{j,m}(x) = \exp(2\pi i j p / c z^m), \tag{17}$$

which shows that some rational multiple of $\mathbb{Z}[1/z]$ is contained in $L_1\sigma_{pp}/2\pi$. □

As a special case, we recover the results of Dekking and Keane, and then of Berend and Radin, since their substitution σ has $n = 4, n_a = 2, n_b = 1$, so $z = 1$. When $L_1 = 2$ and $L_2 = 1$, the point spectrum is $2\pi\mathbb{Z}[1/z] = 2\pi\mathbb{Z}$. This also illustrates the sharp contrast between substitution tiling spaces admitting a flow with point spectrum and spaces admitting equicontinuous flows. If X admits an equicontinuous $\mathbb{R}$ action ϕ with point spectrum σ_{pp} and if ϕ' is any $\mathbb{R}$ action on X obtained by a time change of ϕ, then the point spectrum of ϕ' is $c \cdot \sigma_{pp}$ for some possibly 0 constant c, see Egawa [E]. In the substitution σ currently under consideration, the $(1,1)$ flow has purely point spectrum σ_{pp} but admits a time change represented by $(2,1)$ which has discrete spectrum which can only be represented as a proper subset of a multiple of σ_{pp}.

What about the Fibonacci tiling? In that case the smaller eigenvalue is $1-\tau$, whose magnitude is less than 1, so Theorem 2.4 does not apply directly. However, there is still something that can be said about such cases:

Theorem 2.6. *Suppose that there exists a full recurrence vector. Let S be the span of the (generalized left-) eigenspaces of M with eigenvalue of magnitude strictly less than 1. If k is in the point spectrum, then $kL/2\pi$ is the sum of a rational vector and an element of S.*

Proof. As before, $t_m = LM^m v$ converges to zero (mod 1), so we can write $t_m = i_m + \epsilon_m$ with i_m integral and ϵ_m converging to zero, and with both the i_m's and the ϵ_m's eventually satisfying the recursion relation (6). Since the ϵ_m's converge to zero, they must eventually be linear combinations of m-th powers of the small eigenvalues of M (possibly times polynomials in m, if M is not diagonalizable). By adding an element of S to L (and hence to $kL/2\pi$), we can then get all the ϵ_m's to be identically zero beyond a certain point.

If M is invertible, the resulting value of $kL/2\pi$ must then be rational by the same argument as in the proof of Theorem 2.4. If M is not invertible, then we need an extra step, since the vectors $M^m v, M^{m+1}v, \ldots, M^{m+n-1}v$ are no longer linearly independent. Let $S_0 \subset S$ be the generalized left-eigenspace with eigenvalue zero, and suppose that this space has dimension ℓ. The vectors $M^m v, \ldots, M^{m+n-\ell-1}$ are linearly independent and span the annihilator of S_0, which can be identified with the dual space of $\mathbb{R}^{n*}/S_0$. From the integers $t_m, \ldots, t_{m+n=\ell-1}$ we can reconstruct, by rational operations, a representative of $kL/2\pi$ in $\mathbb{R}^{n*}/S_0$. But that implies that $kL/2\pi$ is a rational vector plus an element of S_0. □

Another way of stating the same result is to say that $kL/2\pi$, projected onto the span of the large eigenvectors, equals the projection of a rational vector onto this span.

This theorem can be used in two different ways. First, it constrains the set of vectors L for which the system admits point spectrum. Let d_b be the number of large eigenvalues, counted with (algebraic) multiplicity. There are only a countable number of possible values for the projection of $kL/2\pi$ onto the span of the large (generalized) eigenvectors. If the projection of L does not lie in the ray generated by one of these points, there is no point spectrum. In other words, one must tune $d_b - 1$ parameters to a countable number of possible values in order to achieve a nontrivial point spectrum.

A second usage is to constrain the spectrum for fixed L. The rational points in $\mathbb{R}^n$, projected onto the span of the large eigenvalues, and then intersected with the ray defined by a fixed L, form a vector space over $\mathbb{Q}$ of dimension at most $n+1-d_b$. As a result, the point spectrum tensored with $\mathbb{Q}$ is a vector space over $\mathbb{Q}$ whose dimension is bounded by one plus the number of small eigenvalues. Below we derive an even stronger result, in which only the small eigenvalues that are conjugate to the Perron-Frobenius eigenvalue contribute to the complexity of the spectrum.

So far we have assumed that there exists a full recurrence vector v. This may fail, either because there is something peculiar about the available recurrence vectors, or because the matrix M has repeated eigenvalues. In such cases we can repeat our analysis on a proper subspace of $\mathbb{R}^n$. There are necessarily fewer constraints on the vector L, but for fixed L the constraints on k are actually stronger than before.

Theorem 2.7. *Let b_{PF} be the number of large eigenvalues that are algebraically conjugate to the Perron-Frobenius eigenvalue λ_{PF} (including λ_{PF} itself), and let s_{PF} be the number of small eigenvalues conjugate to λ_{PF}. For the system to have nontrivial point spectrum, L must lie in a countable union of subspaces of $\mathbb{R}^n$, each of codimension $b_{PF}-1$ or greater. For fixed L, the dimension over $\mathbb{Q}$ of the point spectrum tensored with $\mathbb{Q}$ is at most $s_{PF}+1$.*

Proof. As a first step we diagonalize M over the rationals as far as possible. By rational operations we can always put M in block-diagonal form, where the characteristic polynomial of each block is a power of an irreducible polynomial. Since the Perron eigenvalue λ_{PF} has algebraic multiplicity one, every eigenvalue algebraically conjugate to λ_{PF} also has multiplicity one. Thus there is a unique block whose characteristic polynomial has λ_{PF} for a root. Every recurrence vector is integral, and pairs nontrivially with L_0, and so pairs nontrivially with every left-eigenvector of this block. We then consider the constraints on the spectrum that can be obtained from this block alone.

We repeat the argument of the previous two theorems. As before, we modify L by adding a linear combination of small eigenvectors so as to make t_m eventually integral. Since the block diagonalization was over the rationals, the vector $\vec{t} = (t_m, \dots, t_{m+n-1})^T$, expressed in the new basis, is still rational. From this vector we deduce, as before, the projection of $kL/(2\pi)$ on any large eigenvector with which our recurrence vector v pairs nontrivially. This gives at least b_{PF} independent constraints on the pair (L, k), hence $b_{PF} - 1$ constraints on L if the spectrum is to be nontrivial.

Next we consider the projection of $kL/(2\pi)$ onto the large eigenspaces of the Perron-Frobenius block. Since only $b_{PF} + s_{PF}$ components of $\vec{t}$ (expressed in the new basis) contribute, this is the projection of $\mathbb{Q}^{b_{PF}+s_{PF}}$ onto $\mathbb{R}^{b_{PF}}$, whose real span is all of $\mathbb{R}^{b_{PF}}$.

Intersected with the ray defined by a fixed L, this gives a vector space of dimension at most $s_{PF}+1$ in which k can live. □

3. Topological conjugacies

In Section 2 we studied the point spectrum of a tiling space. This is a conjugacy invariant; systems with different discrete spectra cannot be conjugate. However, we found many different values of L that gave exactly the same point spectrum – namely $\{0\}$. In this section we give criteria for determining when two values of L give rise to conjugate tiling spaces. We supply sufficient conditions for conjugacy in terms of the matrix M. We also supply necessary conditions; these require knowing something about the actual substitution, and not just its matrix M. Finally, we exhibit examples that demonstrate the need for such details. Different substitutions with the same matrix may result in different criteria for conjugacy.

Throughout this section, $\mathcal{S}$ is a substitution subshift with substitution map σ and substitution matrix M and we consider two possible length vectors $L = (f(a_1), f(a_2), \dots)$ and $L' = (g(a_1), g(a_2), \dots)$. The substitution applied ℓ times to a single tile (or letter) is called a *supertile* of order ℓ.

Theorem 3.1. *If, for some integer k,*

$$\lim_{m\to\infty} \left(LM^{m+k} - L'M^m\right) = 0, \tag{18}$$

then $\mathcal{T}_f$ is conjugate to $\mathcal{T}_g$.

Proof. We prove the theorem in two steps. First we prove it in the special case that $k=0$, and then in the case that $L = L'M$. Combining these results then gives the theorem for all $k \le 0$. The situation for $k>0$ is the same, only with $\mathcal{T}_f$ and $\mathcal{T}_g$ reversed.

If (18) holds with $k=0$, then the supertiles in the $\mathcal{T}_f$ system asymptotically have the same size as the corresponding supertiles in the $\mathcal{T}_g$ system, and the convergence is exponential. We then adapt the argument that Radin and Sadun [RS] applied to the Fibonacci tiling. If x is a tiling in $\mathcal{T}_f$ with tiles in a sequence $u = \dots u_{-1}u_0u_1, \dots$, then we require $\phi_\ell(x) \in \mathcal{T}_g$ to be a tiling with the exact same sequence of tiles. The only question is where to place the origin. Let d_ℓ be the coordinate of the right edge of the order-ℓ supertile of x that contains the origin, and let $-e_\ell$ be the coordinate of the left edge. Place the origin in $\phi_\ell(x)$ a fraction $e_\ell/(d_\ell+e_\ell)$ of the way across the corresponding supertile of order ℓ in the $\mathcal{T}_g$ system. (This is essentially the map h of Theorem 1.1, only applied to supertiles of order ℓ.) Since the sizes of the supertiles converge exponentially, the location of the origin in $\phi_\ell(x)$ converges, and we can define $\phi(x) = \lim_{\ell\to\infty} \phi_\ell(x)$. All that remains is to show that ϕ is a conjugacy. The approximate map ϕ_ℓ is not a conjugacy; translations in x the keep the origin in the same supertile of order ℓ get magnified (in $\phi(x)$) by the ratio of the sizes of the supertiles of order ℓ containing the origin in x and $\phi(x)$. However, this ratio goes to one as $\ell \to \infty$, while the range of translations to which this ratio applies grows exponentially. In the $\ell \to \infty$ limit, ϕ commutes with translation by any $s \in (-e_\infty, d_\infty)$. Typically this is everything. The case where x contains two infinite-order supertiles is only slightly trickier – once one sees that ϕ preserves the boundary between these infinite-order supertiles, it is clear that ϕ commutes with all translations.

Now suppose that $L = L'M$. Then the tiles in the $\mathcal{T}_f$ system have exactly the same size as the corresponding supertiles of order 1 in the $\mathcal{T}_g$ system. We define our conjugacy ϕ as follows: If x is a tiling in $\mathcal{T}_f$ with tiles in a sequence $u = \ldots u_{-1}u_0u_1\ldots$, then $\phi(x)$ to a tiling in $\mathcal{T}_g$ with sequence $\sigma(u)$, with each tile in x aligned with the corresponding order-1 supertile in $\phi(x)$.

Combining the cases, we obtain conjugacy whenever (18) applies. □

Theorem 3.1 gives sufficient conditions for two tiling spaces to be conjugate. To derive necessary conditions we must understand the extent to which words in our subshift $\mathcal{S}$ (and its associated tiling spaces) can repeat themselves. Mossé shows in [M1] that in every substitution subshift there exists an integer N so that no word is ever repeated N or more times. We will need to refine these results by considering words that are repeated a fractional number of times. Counting fractional periods depends on the length vector: the word *ababa* has its basic period *ab* repeated $2 + f(a)/(f(a) + f(b))$ times.

Definition. A recurrence vector v is a *repetition vector of degree* p of a tiling space $\mathcal{T}$ if there are finite words w (with population vector v) and w', such that: 1) w' contains w, 2) w' is periodic with period equal to the length of w, 3) the length of w' is at least p times the length of w, and 4) w' appears in some (and therefore every) tiling in $\mathcal{T}$.

Note that every repetition vector of degree $p > p'$ is also a repetition vector of degree p'. Note also that the word w is typically not uniquely defined by v. A cyclic permutation of the tiles in w typically yields a word that works as well. For instance, if $w' = ababa$ appears in a tiling, then $v = (1, 1)^T$ is a repetition vector with period $2 + f(a)/(f(a) + f(b))$, and we may take either $w = ab$ or $w = ba$.

We denote the length L_0w of a word w in $\mathcal{T}_0$ as described above by $|w|$. If v is a repetition vector of degree p for $\mathcal{T}_0$, representing the word w sitting inside w' in $u \in \mathcal{A}^{\mathbb{Z}}$, then Mv is a repetition vector of degree p, representing the word $\sigma(w)$ sitting inside $\sigma(w')$ in $\sigma(u) \in \mathcal{A}^{\mathbb{Z}}$. The degree is the same, since in $\mathcal{T}_0$ the substitution σ stretches each word by exactly the same factor, namely λ_{PF}. Thus, every repetition vector v gives rise to an infinite family of repetition vectors M^kv. The following theorem limits the number of such families.

Theorem 3.2. *Let $p > 1$. There is a finite collection of vectors $\{v_1, \ldots, v_N\}$ such that every repetition vector of degree p for $\mathcal{T}_0$ is of the form M^kv_i for some pair (k, i).*

Proof. There is a recognition length $D_{\mathcal{S}}$ of the subshift $\mathcal{S}$ such that knowing a letter, its $D_{\mathcal{S}}$ immediate predecessors and its $D_{\mathcal{S}}$ immediate successors determines the supertile of order 1 containing that letter, and the position within that supertile of the letter. Thus for every substitution tiling space $\mathcal{T}$ there is a recognition length $D_{\mathcal{T}}$ such that the neighborhood of radius $D_{\mathcal{T}}$ about a point determines the supertile of order 1 containing that point, and the position of that point within the supertile. (E.g., one can take $D_{\mathcal{T}}$ to be $1 + D_{\mathcal{S}}$ times the size of the largest tile.) Let $D = D_{\mathcal{T}_0}$.

Suppose v is a repetition vector of degree p for $\mathcal{T}_0$, corresponding to a word w that is repeated p times in a word w', whose length is much greater than D. Then there is an interval of size $|w'| - 2D$ in which the supertiles of order 1 are periodic with period $|w|$. Thus v is the population vector of a collection of supertiles of order 1, so there exists a population vector v_1 of a word w_1, such that $v = Mv_1$, and such that v_1 is a repetition vector of degree $(|w'| - 2D)/|w| = p - 2D/\lambda_{PF}|w_1|$. Note that w itself does not have to equal $\sigma(w_1)$ — it may happen that w is a cyclic permutation of $\sigma(w_1)$.

Repeating the process, we find words w_i such that $\sigma(w_i)$ equals w_{i-1} (up to cyclic permutation of tiles), and such that the population vector of w_i is a repetition vector of degree

$$p - \frac{2D}{|w_i|}(\lambda_{PF}^{-1} + \lambda_{PF}^{-2} + \cdots + \lambda_{PF}^{-i}) \geq p - \frac{2D}{|w_i|}\sum_{\ell=1}^{\infty} \lambda_{PF}^{-\ell} = p - \frac{2D}{|w_i|(\lambda_{PF} - 1)}. \tag{19}$$

Now pick $\epsilon < p - 1$. We have shown that every repetition vector of degree p is of the form $M^k v_i$, where v_i is a repetition vector of degree $p - \epsilon$, and where $L_0 v_i$ is bounded by $\frac{2D\lambda_{PF}}{\epsilon(\lambda_{PF}-1)}$. However, there are only a finite number of words of this length, hence only a finite number of possible repetition vectors v_i. □

As defined, the degree of a repetition vector depends on the length vector of the tiling space. This difference between tiling spaces disappears in the limit of long repetition vectors:

Lemma 3.3. *Let L be decomposed as $L = cL_0 + L_r$, where L_r is a linear combination of (generalized) eigenvectors with eigenvalue less than λ_{PF}. For each word w, let r_w be the ratio of the length of w in $\mathcal{T}_f$ to the length of w in $\mathcal{T}_0$. There exist constants $c_{1,2} > 0$ such that $|r_w - c| < c_1/|w|^{c_2}$.*

The proof is trivial. □

Lemma 3.4. *Let $\epsilon > 0$, and let L and L' be fixed. There is a length R such that, if v is a repetition vector of degree p in $\mathcal{T}_f$, representing a word w of length greater than R, then v is a repetition vector of degree $p - \epsilon$ in $\mathcal{T}_g$.*

Again, trivial. □

Pick $p_0 > 1$ such that there exist repetition vectors of degree greater than p_0 in $\mathcal{T}_0$. By Theorem 3.2, there are a finite number of vectors v_i such that every repetition vector of degree p_0 is of the form $M^k v_i$. Now, for each k, i, let $p_{k,i}$ be the maximal degree of $M^k v_i$, and let $p_i = \lim_{k\to\infty} p_{k,i}$. Since the p_i's are a finite set of real numbers, all greater than p_0, there are real numbers p and ϵ such that $p > p_0 + \epsilon$ and such that each p_i is greater than $p + \epsilon$. Let L, L' and L_0 be given. By lemma 3.4, there is a length R such that the set of repetition vectors v of degree p and with $L_0 v > R$ is precisely the same for the three tiling systems $\mathcal{T}_f$, $\mathcal{T}_g$ and $\mathcal{T}_0$. Pick generators $v_1, \ldots, v_N$ of the finite families of these repetition vectors, with $L_0 v_i \leq L_0 v_{i+1}$, and with $L_0 v_N < \lambda_{PF} L_0 v_1$.

The asymptotic lengths of these repetition vectors are conjugacy invariants:

Theorem 3.5. *Suppose that $\mathcal{T}_f$ and $\mathcal{T}_g$ are conjugate and that $\{v_1, \ldots v_N\}$ is a collection of repetition vectors of degree p that generates all repetition vectors of degree p. Then,* *given* $\delta > 0$*, for every* $i \in \{1, \dots, N\}$ *and for any sufficiently large integer* m*, there exist* j, m' *such that* $|LM^m v_i - L' M^{m'} v_j| < \delta$.

Proof. Let $\phi : \mathcal{T}_f \to \mathcal{T}_g$ be the conjugacy. Since $\mathcal{T}_f$ and $\mathcal{T}_g$ are compact metric spaces, ϕ is uniformly continuous. Now suppose v_i is a repetition vector of degree p. Let $t_{m,i} = LM^m v_i$. Then for every tiling $x \in \mathcal{T}_f$ there is a range of real numbers r of size roughly $(p-1)t_{m,i}$, such that $T_r(x)$ is very close to $T_{r+t_{m,i}}(x)$, where T_r denotes translation by r. Specifically, as $m \to \infty$, the distance from $T_{r+t_{m,i}}(x)$ to $T_r(x)$ can be made arbitrarily small, and the size of the range of r's, divided by $t_{m,i}$, can be made arbitrarily close to $p-1$. This implies that $\phi(T_r(x))$ and $\phi(T_{r+t_{m,i}}(x))$ are extremely close. But $\phi(T_r(x)) = T_r(\phi(x))$ and $\phi(T_{r+t_{m,i}}(x)) = T_{r+t_{m,i}}(\phi(x))$, so $\mathcal{T}_g$ admits a repetition vector whose pairing with L' is extremely close to $t_{m,i}$, and whose degree is extremely close to p. However, for m large, all such repetition vectors are of the form $M^{m'} v_j$. □

As an illustration of the power of this theorem, consider the substitution $a \to aaaabb$, $b \to babbba$. The substitution matrix is $\left(\begin{smallmatrix} 4 & 2 \\ 2 & 4 \end{smallmatrix}\right)$, whose eigenvalues, namely 6 and 2, are both greater than one. It is easy to see that $v = (1,0)^T$ is a repetition vector of degree 5, and this vector generates the only family of repetition vectors with $p = 5$. If $\mathcal{T}_f$ and $\mathcal{T}_g$ are conjugate, for each large m there must exist m' such that $(LM^m - L'M^{m'})v$ is small. Since v is full, this implies there is an integer k such that $L' = LM^k$. Those are the only possible conjugacies.

As a second example, consider the substitution $a \to aaaabb$, $b \to bbbbaa$. This has exactly the same substitution matrix, but the periodicities are different. $v_1 = (1,0)^T$ and $v_2 = (0,1)^T$ are both repetition vectors with degree 6. The possibility of having $j \neq i$ in Theorem 3.5 allows for additional conjugacies. Indeed, it is not hard to see that $\mathcal{T}_f$ and $\mathcal{T}_g$ are conjugate if (and only if) either $L' = (L_1, L_2)M^k$ or $L' = (L_2, L_1)M^k$ for some (possibly negative) integer k.

The difference between these two examples illustrates that one cannot obtain sharp conditions on conjugacy from the substitution matrix alone. One needs some details about the substitution itself.

Another class of examples illustrating the importance of the actual substitution as opposed to its matrix can be constructed using powers of matrices. For instance, the substitution $a \to aabaabbba$, $b \to bbabbaaab$ has matrix $M = \left(\begin{smallmatrix} 5 & 4 \\ 4 & 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)^2$ and is the square of the substitution $a \to aab$, $b \to bba$, and so it can be seen that the associated tiling space admits (among others) conjugacies with $L' = L \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)^m$, where m can be any integer. But the substitution $a \to aaaaabbbb$, $b \to abbbbbaaa$, with the same matrix M, has only one family of repetition vectors of degree 8, namely those generated by $(1,0)^T$. In this case all conjugacies are of the form $L' = LM^k = L \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)^{2k}$.

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Alex Clark: Department of Mathematics, University of North Texas, Denton, Texas 76203

E-mail address: AlexC@unt.edu

Lorenzo Sadun: Department of Mathematics, The University of Texas at Austin, Austin, TX 78712-1082 U.S.A.

E-mail address: sadun@math.utexas.edu