The Periodic Multidimensional Schrödinger Operator, Part 4, An Algorithm for Finding the Potential by Spectral Invariants

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Abstract

In this paper, we give an algorithm for finding the potential of the three-dimensional Schrödinger operator by spectral invariants.

1 Introduction

We investigate the Schrödinger Operator

$$L(q) = -\Delta + q(x),\ x \in R^d,\ d \geq 2$$

with a real periodic (relative to the Lattice Ω) potential $q(x) \in W_2^l(F)$, where $l \geq 6(3^d(d+1)^2) + d$, and F is the fundamental domain R^d/Ω of Ω. The spectrum of $L(q)$ is the union of the spectra of the operators $L_t(q)$, for $t \in F^* \equiv R^d/\Gamma$ generated by $-\Delta u + q(x)u$ and the conditions (see [1])

$$u(x+\omega) = e^{i\langle t,\omega\rangle}u(x),\ \forall\omega \in \Omega,$$

where $\Gamma \equiv \{\delta \in R^d : \langle \delta, \omega\rangle \in 2\pi\mathbb{Z}, \forall\omega \in \Omega\}$ is the lattice dual to Ω and $\langle .,. \rangle$ is the inner product in R^d. The eigenvalues $\Lambda_1(t) \leq \Lambda_2(t) \leq ...$of L_t define functions $\Lambda_n(t)$ $(n = 1, 2, ..)$ of t that are called the band functions or Floquet spectrum of L.

In paper [5] by the Floquet spectrum of L we constructively determined the following spectral invariants

$$\int_F |q^a(x)|^2\, dx, \tag{1}$$

$$\int_F |q_{a,\beta}(x)|^2\, q^a(x) dx, \tag{2}$$

where a is the maximal element of Γ, that is, a is the element of Γ of minimal norm belonging to the line aR, the $q^a(x) = \sum_{n\in Z} q_{n\delta} e^{in\langle a,x\rangle}$ is directional (one dimensional) potential $q_b \equiv (q(x), e^{i\langle b,x\rangle})$ is the Fourier coefficient of $q(x)$. We assume that $q_0 = 0$. The function $q_{a,\beta}(x)$ is defined as

$$q_{a,\beta}(x) = \sum_{b\in S(a,\beta)\backslash aR} \frac{b}{\langle \beta, b\rangle} q_b e^{i\langle b,x\rangle},$$

where $S(a,\beta) = P(a,\beta)\cap\Gamma$, and $P(a,\beta)$ is the plane containing a, β and 0. Here $\beta\in\Gamma_a$, where Γ_a is the lattice dual to Ω_a and Ω_a is the sublattice $\{d\in\Omega : \langle d,a\rangle = 0\}$ of Ω in the hyperplane $H_a = \{x\in R^d : \langle x,a\rangle = 0\}$. If

$$q^a(x) = q_a e^{i\langle a,x\rangle} + q_{-a} e^{-i\langle a,x\rangle} \tag{3}$$

then besides (1), (2) we found the invariant

$$\int_F |q_{a,\beta}(x)|^2 (q_a^2 e^{i2\langle a,x\rangle} + q_{-a}^2 e^{-i2\langle a,x\rangle}) dx. \tag{4}$$

In this paper fixing the translations $q(x)\to q(-x),\ q(x)\to q(x+\tau),\ \tau\in R^3$, we give an algorithm for finding the potential $q(x)$ ot the three- dimensional Schrödinger operator $L(q)$ by the invariants (1), (2) and (4).

Note that the operators $L(q(x)),\ L(q(-x)),\ L(q(x+\tau))$ have the same band functions, i.e., $q(x),\ q(-x),\ q(x+\tau)$ are isospectral potentials. Eskin *et al.* [2], [3] proved the following result for the two-dimensional periodic Schrödinger operator:

For $\Omega\subset R^2$ satisfying the condition: if $|\omega| = |\omega'|$ for $\omega,\ \omega'\in\Omega$, then $\omega' = \pm\omega$; there is a set $\{M_\alpha\}$ of manifolds of potentials such that

a) $\{M_\alpha\}$ is dense in the set of smooth periodic potentials in $\mathbb{C}^\infty-$topology;

b) for each α there is a dense open set $Q_\alpha \subset M_\alpha$ such that for $q \in Q_\alpha$ the set of real analytic $\widetilde{q}$ satisfying $Spec(L_0(q) = Spec(L_0(\widetilde{q}))$

and the set of $\widetilde{q} \in \mathbb{C}^6(R^n/L)$ satisfying $Spec(L_t(q) = Spec(L_t(\widetilde{q})) \ \forall t \in R^n$ are finite modulo translations.

For the case of the two-dimensional difference periodic Schrödinger operator Gieseker *et al.*[4] proved that:

There is a Zariski-open and dense subset P of L^2 such that for any $V \in P$, $V' \in L^2$ equality $B(V) = B(V')$ of the Bloch varieties implies that V and V' coincide up to symmetries, where $L^2 = L^2(\mathbb{Z}^2/a\mathbb{Z} \oplus b\mathbb{Z})$, a and b are distinct odd primes,

$$B(V) = \{(\xi_1, \xi_2, \lambda) \in \mathbb{C}^\star \times \mathbb{C}^\star \times \mathbb{C} \mid there\ is\ a\ complex-valued$$

$$function\ \Psi\ on\ \mathbb{Z}^2\ satisfying$$

$$\Psi(m+1,n) + \Psi(m-1,n) + \Psi(m,n+1) + \Psi(m,n-1) + V(m,n)\Psi(m,n) =$$

$$\lambda\Psi(m,n)\ and\ \Psi(m+a,n) = \xi_1\Psi(m,n),$$

$$\Psi(m,n+b) = \xi_2\Psi(m,n)\ for\ all\ (m,n)\ in\ \mathbb{Z}^2\}.$$

In Section 3 and 4 we give an algorithm for finding the potential $q(x)$ by the invariants (1), (2) and (4) under the assumption that $q(x)$ is a trigonometric polynomial. In Theorem 10 we construct a dense in $L_2(F)$ set E_0 of trigonometric polynomials, such that every $q \in E_0$ can be found by this algorithm. In Theorem 11 we give a scheme for construction of smooth potential. Then we give an application of Theorem 11 (see Corollary).

2 On Invariants for Polynomials

We assume that $q(x)$ is a trigonometric polynomial of the following form

$$q(x) = \sum_{a \in Q_{N,M,S}} q_a e^{i\langle a,x\rangle}, \quad q_a \neq 0, \tag{5}$$

where $Q_{N,M,S} = \{n\gamma_1 + m\gamma_2 + s\gamma_3 : |n| \leq N, \ |m| \leq M, \ |s| \leq S\}\backslash\{0\}$ for some basis $\gamma_1, \gamma_2, \gamma_3$ of Γ. Here N, M and S are prime numbers satisfying

$$S > 3M + 2, \quad M > 2N, \quad N \gg 1. \tag{6}$$

We introduce the following notation

$$q_a = z(a) = x(a) + iy(a) = r(a)e^{i\alpha(a)}, \quad \alpha(a) \in [-\pi, \pi).$$

If $a = n\gamma_1 + m\gamma_2 + s\gamma_3$, then sometimes we shall write (n, m, s), $z(n, m, s)$ instead of a and $z(a)$ respectively. We say that the Fourier coefficient $z(n, m, s)$ corresponds to (n, m, s). For simplicity, we write Q instead of $Q_{N,M,S}$.

Definition 1. *A maximal vector $a \in \Gamma$ is said to be long maximal (with respect to Q) if $a \in Q$, but $2a \notin Q$, i.e., $sa \in Q$ if and only if $s = \mp 1$.*

If a is long maximal, then the directional potential q^a has form (3). Therefore the invariant (1) is

$$\|q^a\|^2 \equiv 2|q_a|^2. \tag{7}$$

Let us consider the invariants (2) and (4) in this case. Let $\beta \in \Gamma_a\backslash 0$ and $Q(a, \beta) = Q \cap P(a, \beta)$. Putting the Fourier decompositions (3) of directional potentials $q^a(x)$ into (2), (4) and using the introduced notation, we get the invariants

$$\mathbf{Re}z(-a)\Big(\sum_{b\in Q(a,\beta)\backslash aR} \frac{\langle a-b, b\rangle}{-(\langle b, \beta\rangle)^2} z(a-b)z(b)\Big), \tag{8}$$

$$\mathbf{Re}z^2(-a)\Big(\sum_{b} \frac{\langle a+b, a-b\rangle}{-(\langle b, \beta\rangle)^2} z(a+b)z(a-b)\Big), \tag{9}$$

where the last sum is taken under the conditions $(a+b)$, $(a-b) \in Q(a, \beta)\backslash aR$. Here $\langle b, \beta\rangle \neq 0$, since the line orthogonal to β in plane $P(a, \beta)$ is aR, but $b \notin aR$. If $b \in \Gamma \cap P(a, \beta)$ then the plane $P(a, \beta)$ coincides with the plane $P(a, b)$. Moreover by (12) of [5] every vector $b \in \Gamma$ has decomposition $b = \beta + ca$, where $\beta \in \Gamma_a, c \in R$. Hence for every plane $P(a, b)$, where $b \in \Gamma$, there exists the plane $P(a, \beta)$, where $\beta \in \Gamma_a$, which coincides with $P(a, b)$. Therefore for each long maximal $a \in \Gamma$ and for each plane $P(a, b)$, where $b \in \Gamma$, we have the invariants (8), (9).

Lemma 2. *Let a be a long maximal element of Q. If the plane $P(a,b)$, where $b \in \Gamma$, does not contain any vector c except $\pm b$ and $\pm(a-b)$, such that $c,\ (a-c) \in Q \backslash aR,\ \langle c,\ a-c\rangle \neq 0$, then the invariant (8) for a and the plane $P(a,b)$ has form*

$$\mathbf{Re} z(-a)z(a-b)z(b) \tag{10}$$

Lemma 3. *If a is long maximal and the plane $P(a,b)$, where $b \in \Gamma$, does not contain any vector c except $\pm b$, such that $c,\ (a-c) \in Q \backslash aR,\ \langle c,\ a-c\rangle \neq 0$, then the invariant (9) for a and the plane $P(a,b)$ has form*

$$\mathbf{Re} z^2(-a)z(a+b)z(a-b). \tag{11}$$

Finding the planes satisfying the conditions of these lemmas and using the simple invariants (7), (10), (11), we shall determine the Fourier coefficients $z(a)$ of polynomial (5) in Sections 3 and 4.

3 Finding the Fourier Coefficients Corresponding to the Boundary Points of Q

In this section, we find the Fourier coefficients $z(n,m,s)$ if either $n = N, -N$, or $m = M, -M$, or $s = S, -S$ using the following two observations:

1) All boundary points $a = (n,m,s)$ except points $(\pm N, 0, 0)$, $(0, \pm M, 0)$, $(0, 0, \pm S)$ and cases $|n| = |m| = |s| = N$, are long maximal, since N, M, S are distinct prime numbers. Hence, $r(a)$ is known by (7).

2) If a is a boundary point of Q, then there are a lot of vectors b such that the plane $P(a,b)$ satisfies the conditions of Lemma 2. That is, we have the simple invariant (11). So from (7) and (11), we get the value of

$cos(-2\alpha(a)+\alpha(a+b)+\alpha(a-b))$. In other words, for all a and b satisfying the conditions of Lemma 2, we have the equation

$$-2\alpha(a) + \alpha(a+b) + \alpha(a-b) = d(a,b)e(a,b)(mod 2\pi), \tag{12}$$

where $e(a,b)$ is known number in $[0,\pi]$, $d(a,b)$ is either 1 or -1 and $c = d(mod 2\pi)$ means that $d - c = 2k\pi$ for some integer k. For simplicity of notation of $e(a,b)$ and $d(a,b)$ we shall write e_i, d_i. For different pairs (a,b) we write different indices (different numbers 1,2,.., as well as, different letters $i, j, l, ...$). So we can write a lot of linear equations with respect to the argument of Fourier coefficients. If we know the sign of $d(a,b)$ and the values of two summands in the left side of (12), then we can find the value of third term in the left side of this equation. To use these equations we need to know the values of argument of some Fourier coefficients. Three of them can be determined by fixing the translation $q(x) \to q(x+\tau)$. It means that we take one of the functions $q(x+\tau)$, i.e., we fix one value of τ in F. The conditions

$$\alpha_\tau(N-1,M,S) = \alpha_\tau(N,M-1,S) = \alpha_\tau(N,M,S-1) = 0, \tag{13}$$

$$\alpha_\tau(N,M,S) \in [0, \frac{2\pi}{N+M+S-1}), \tag{14}$$

where $\alpha_\tau(a) = \arg(q(x+\tau),\ e^{i\langle a,x\rangle}) = \langle a,\tau\rangle + \alpha(a)$, determine a unique value of τ. Indeed, every $\tau \in F$ has the decomposition $\tau = c_1\omega_1 + c_2\omega_2 + c_3\omega_3$; $c_1, c_2, c_3 \in [0,1)$ and the conditions (13) are equivalent to

$$2\pi((N-1)c_1 + Mc_2 + Sc_3) = -\alpha(N-1,M,S)(mod 2\pi),$$

$$2\pi(Nc_1 + (M-1)c_2 + Sc_3) = -\alpha(N,M-1,S)(mod 2\pi),$$

$$2\pi(Nc_1 + Mc_2 + (S-1)c_3) = -\alpha(N,M,S-1)(mod 2\pi).$$

This system of equations with respect to the unknowns $c_1,\ c_2,\ c_3 \in [0,1)$ has $N+M+S-1$ solutions $(c_{1,l},\ c_{2,l},\ c_{3,l})$ satisfying

$$c_{j,l+1} - c_{j,l} = \frac{1}{N+M+S-1}, \quad j = 1,2,3. \quad l = 1,2,...,N+M+S-2.$$

Hence we obtain $\tau_1,\ \tau_2, ...,\ \tau_{N+M+S-1}$ such that $\tau_{l+1} - \tau_l = \frac{\omega_1+\omega_2+\omega_3}{N+M+S-1}$. Then

$$\alpha_{\tau_{l+1}}(N,M,S) - \alpha_{\tau_l}(N,M,S) = \frac{2\pi}{N+M+S-1}. \tag{15}$$

This together with (14) gives us the unique value of τ. Thus we can take

$$\alpha(N-1,M,S) = \alpha(N,M-1,S) = \alpha(N,M,S-1) = 0, \tag{16}$$

$$\alpha(N,M,S) \in [0, \frac{2\pi}{N+M+S-1}). \tag{17}$$

On the other hand, the invariants (7) determine the modulus of

$$z(N-1,M,S),\ z(N,M-1,S),\ z(N,M,S-1). \tag{18}$$

So the Fourier coefficients (18) are known. Now using (18), we find all Fourier coefficients $z(n,m,s)$ of $q(x)$ step by step.

Step 1: In this step we fix the translation $q(x)\to q(-x)$ and find

$$z(N,M,s),\ z^2(N,M-1,s),\ z(N,M-2,s)\ \ \forall s, \tag{19}$$

considering the following lemma

Lemma 4. *The plane $P(a,b)$ satisfies the conditions of Lemma 2 in each of the following cases:*

1) $a=(N,M-1,s)$, $b=(0,\pm1,p)$, where $s+p,\ s-p\in[-S,S]$,

2) $a=(N,m,S-1)$, $b=(0,q,\pm1)$, where $m+q,\ m-q\in[-M,M]$,

3) $a=(N,m,s)$, $b=(0,\pm1,p)$, where $m\in[-M+1,M-1]$, $s+p,s-p\in[-S,S]$, but $s-2p\notin[-S,S]$.

Proof: Let $c=(n,m,l)$ be any nonzero vector satisfying $(a+c),\ (a-c)\in Q$. By definition of Q, this condition implies that $n=0$ for all cases 1-3. Hence c belongs to intersection of planes $P(a,b)$ and $\{x_1=0\}\equiv\{(x_1,x_2,x_3)\in R^3 : x_1=0\}$. It means that, $c=kb$ for some nonzero integer k. Clearly, if k is not ±1, then either $a+c$ or $a-c$ is not in Q in all cases ■

¿From Lemma 2 and Lemma 3, it follows that we have the equation (12) for $a=(N,M-1,s)$, $b=(0,1,p)$. By taking $p=1,0,-1$, we get the equations

$$-2\alpha(N,M-1,s)+\alpha(N,M,s+1)+\alpha(N,M-2,s-1)=d_ie_i(mod2\pi),$$

$$-2\alpha(N,M-1,s)+\alpha(N,M,s)+\alpha(N,M-2,s)=d_je_j(mod2\pi), \tag{20}$$

$$-2\alpha(N,M-1,s)+\alpha(N,M,s-1)+\alpha(N,M-2,s+1)=d_ke_k(mod2\pi).$$

Suppose the constants d_i, d_j, d_k are already determined. Then from these equations, we can find

$$-2\alpha(N,M-1,l),\ \alpha(N,M,l),\ \alpha(N,M-2,l)\ \text{ for }\ l=s-1, \tag{21}$$

if (21) are known for $l = s+1, s$. We find (21) for $l = S, S-1, S-2$ as follows. Substituting $a = (N, M, S-1),\ b = (0,0,1);\ a = (N, M-1, S),\ b = (0,1,0);$

$$a = (N, M-1, S-1),\ b = (0,1,-1);\ a = (N, M-1, S-1),\ b = (0,1,0);$$

$$a = (N, M-1, S-1),\ b = (0,0,1);\ a = (N, M-1, S-1),\ b = (0,1,1)$$

into (12) and using (16) we obtain the equations

$$\alpha + \alpha(N, M, S-2) = d_1 e_1 (mod 2\pi),$$

$$\alpha + \alpha(N, M-2, S) = d_2 e_2 (mod 2\pi),$$

$$-2\alpha(N, M-1, S-1) + \alpha(N, M, S-2) + \alpha(N, M-2, S) = d_3 e_3 (mod 2\pi),$$

$$-2\alpha(N, M-1, S-1) + \alpha(N, M-2, S-1) = d_4 e_4 (mod 2\pi),$$

$$-2\alpha(N, M-1, S-1) + \alpha(N, M-1, S-2) = d_5 e_5 (mod 2\pi),$$

$$-2\alpha(N, M-1, S-1) + \alpha + \alpha(N, M-2, S-2) = d_6 e_6 (mod 2\pi),$$

where we denote the $\alpha(N, M, S)$ by α and for a while consider it as known. Here we have used the fact that the planes $P(a, b)$ satisfy the conditions of Lemma 2 for all above listed values of a and b (see Lemma 3). From these equations, we obtain

$$\alpha(N, M, S-2) = (d_1 e_1 - \alpha)(mod 2\pi),$$

$$\alpha(N, M-2, S) = (d_2 e_2 - \alpha)(mod 2\pi),$$

$$-2\alpha(N, M-1, S-1) = (d_3 e_3 - d_2 e_2 - d_1 e_1 + 2\alpha)(mod 2\pi),$$

$$\alpha(N, M-2, S-1) = (d_4 e_4 + d_2 e_2 + d_1 e_1 - d_3 e_3 - 2\alpha)(mod 2\pi),$$

$$\alpha(N, M-1, S-2) = (d_5e_5 + d_2e_2 + d_1e_1 - d_3e_3 - 2\alpha)(mod 2\pi),$$

$$\alpha(N, M-2, S-2) = (d_6e_6 + d_2e_2 + d_1e_1 - d_3e_3 - 3\alpha)(mod 2\pi).$$

Combining these with the equations

$$-2\alpha(N, M-2, S-1) + \alpha(N, M-2, S) + \alpha(N, M-2, S-2) = d_7e_7(mod 2\pi),$$

$$-2\alpha(N, M-1, S-2) + \alpha(N, M, S-2) + \alpha(N, M-2, S-2)) = d_8e_8(mod 2\pi)$$

obtained from (12) by taking $b = (0,0,1)$, $a = (N, M-2, S-1)$ and $b = (0,1,0)$, $a = (N, M-1, S-2)$, we get

$$f_1(V) = 0(mod 2\pi),\ f_2(U) = d_6e_6 + d_3e_3(mod 2\pi), \tag{22}$$

where $V = (d_1, d_3, d_4, d_6, d_7)$, $f_1(V) \equiv d_7e_7 - (d_6e_6 - 2d_4e_4 + d_3e_3 - d_1e_1)$, $U = (d_8, d_5, d_2)$, $f_2(U) \equiv d_8e_8 + 2d_5e_5 + d_2e_2$ and d_i is either 1 or -1. Hence the vector V takes 32 distinct values $V_1, V_2, ..., V_{16}$ and $-V_1, -V_2, ..., -V_{16}$. Then the function $f_1(V)$ takes 32 values $f_1(V_k)$, $f_1(-V_k)$ $(k = 1, 2, ..., 16)$. Similarly, the vector U takes 8 distinct values $U_1, U_2, ..., U_8$ and the function $f_2(U)$ takes 8 values $f_2(U_k)$ $(k = 1, 2, ..., 8)$.

Assume that, for $k \neq j$

$$f_1(V_k) - f_1(Vj),\quad d_6e_6 + d_3e_3,\ f_2(U_k) - f_2(U_j) \neq 0(mod 2\pi). \tag{23}$$

(Note that these conditions mean that we have the conditions on invariants (11), i.e., corresponding Fourier coefficients.) Then there are only one index k and two values $V_k,\ -V_k$ of V satisfying $f_1(V_k) \equiv -f_1(-V_k) = 0(mod 2\pi)$. On the other hand, the argument of Fourier coefficients of $q(-x)$ takes the opposite value of the argument of Fourier coefficients of $q(x)$. Therefore, for fixing the translation

$$q(x) \longrightarrow q(-x), \tag{24}$$

we take one of these two remaining values $V_k,\ -V_k$ of V. Thus, we have determined the signs of $d_1,\ d_3,\ d_4,\ d_6,\ d_7$ from first inequality in (23). Since the signs of d_3 and d_6 are already known we find d_8, d_5 and d_2 from second and third inequalities in (23). So we found

$$\alpha(N, M-1, S-2) \tag{25}$$

and (21) for $l = S, S-1, S-2$. They satisfy the formulae

$$-2\alpha(N, M-1, S-p) = E_i + 2p\alpha,$$

$$\alpha(N, M, S-p) = E_j - (p-1)\alpha, \tag{26}$$

$$\alpha(N, M-2, S-p) = E_k - (p+1)\alpha$$

for $p = 0, 1, 2$. Further by $Ei,\ Ej, ...$, we denote the linear combinations of e_k with known integer coefficients. These formulae for all p can be easily obtained from (20) by induction. Here the signs of d_i, d_j, d_k (see (20)) can be found as above using the equations

$$-2\alpha(N, M-1, s) + \alpha(N, M, s+2) + \alpha(N, M-2, s-2) = d_n e_n (mod 2\pi),$$

$$-2\alpha(N, M-1, s) + \alpha(N, M, s-2) + \alpha(N, M-2, s+2) = d_m e_m (mod 2\pi)$$

obtained from (12) by taking $b = (0, 1, 2)$, $a = (N, M-1, s)$ and $b = (0, 1, -2)$, $a = (N, M-1, s)$. In the same way, we obtain the formulae

$$\alpha(N, M-p, S) = E_l - (p-1)\alpha, \tag{27}$$

$$\alpha(0, M, -S) = E_q - (2S + N - 1)\alpha. \tag{28}$$

Now let us check that the vector $a = (N, M, 0)$ and the plane $P(a, b)$, where $b = (N, 0, S)$ satisfy the conditions of Lemma 1. For this we prove that this plane contains only the following vectors of Q: 0, $\pm(N, M, 0)$, $\pm(N, 0, S)$, $\pm(0, M, -S)$. In fact, every element (n, m, s) of this plane satisfies the equation

$$S(nM - mN) = sNM. \tag{29}$$

First let us consider the case $s = 0$, i.e., the case $nM = mN$. Since N and M are distinct prime numbers and $-N \leq n \leq N$, $-M \leq m \leq M$, it follows that either $n = \pm N$, $m = \pm M$ or $n = m = 0$. Now we consider the case $s \neq 0$. Then the left side of (29) is multiple of S. Therefore taking into account that S is prime number (see (6)) and $-S \leq s \leq S$, we have $s = \pm S$. This together with (29) gives us the relation $(n \pm N)M = mN$ and arguing as above, we obtain that either $(n = \pm N)$, $m = 0$ or $n = 0$, $m = \pm M$. Hence, using Lemma 1 and formulas (26), (27), (28), from invariants (7), (10) for $a = (N, M, 0)$, $b = (N, 0, S)$ we get the value of

$cos(-\alpha(a) + \alpha(b) + \alpha(a-b)) = cos((N + M + S - 1)\alpha + E_n)$. Similarly, taking $a = (N, M, 0),\ b = (N, 0, -S)$, we find $cos((N+M+S-1)\alpha+E_m)$. By these two values of cos, we determine $(N + M + S - 1)\alpha$ under the condition

$$E_n \neq E_m (mod \pi) \tag{30}$$

This together with (17) gives us the unique value of α and we found the (19).

Step 2: In this step we find

$$z(N, M-1, s) \ \ \forall s. \tag{31}$$

These Fourier coefficients are known up to sign (see (19)), i.e., $z(N, M - 1, s) = d_s e_s$, where e_s is known and d_s is either 1 or -1. Moreover, d_s is known for $s = S, S-2$ (see (16), (25)). To find it for $s = S - 2p$, where $p = 2, 3, ...$, we use the invariant (9) for $a = (N, M-1, S-p)$ and the plane $P(a, b)$, where $b = (0, 0, 1)$. Clearly, if $a + c,\ a - c \in P(a, b) \cap Q$, then c has form $(0, m, s)$. Hence c has to belong to $P(a, b) \cap \{x_1 = 0\} \equiv bR$. It means that $c = (0, 0, q)$ for some integer q. Therefore the invariant (9) for a and the plane $P(a, b)$ has form

$$\mathbf{Re} \sum_q c_q d_{S-p+q} e_{S-p+q} d_{S-p-q} e_{S-p-q} \equiv f_3(V), \tag{32}$$

where c_q are known constants, $V = (h_1, h_2, ..., h_N)$, $h_q = d_{S-p+q} d_{S-p-q}$. Clearly, h_q is either 1 or -1. Assume that the function f_3 takes distinct nonzero values at distinct points V. Then (32) determine h_q and we find $d_{S-p+q} d_{S-p-q}$. Taking $p = q$, we find $d_S d_{S-2p}$. Since d_S is known we find d_{S-2p} and we determine $z(N, M-1, S-2p)$.

To find $z(N, M-1, S-2p+1)$ we use the following two lemmas.

Lemma 5. *The plane $P(a, b)$ satisfies the conditions of Lemma 1 if*

$$a = (N, M-1, s),\ b = (0, -1, N),\ S - N < s \leq S,\ s \neq kN \ \ for \ \ k \in Z.$$

Proof: If the vector $c = (n_1, m_1, s_1)$ lies on plane $P(a, b)$, then the determinant of matrix with rows a, b and c is zero. It gives the condition

$$N(s_1 + m_1 N) = n_1(s + (M-1)N). \tag{33}$$

Since N is prime number and $s + (M - 1)N$ is not multiple of N we have $n_1 = kN$, for $k \in Z$. Then $c = (kN, m_1, s_1)$. The set Q contains the vector c only in the cases $k = 0$, $k = 1$, $k = -1$. In the first case, the condition (33) implies that $s_1 = -Nm_1$, that is $c = m_1(0, -1, N)$. If $m_1 \neq 1$, i.e., $c \neq b$, then the conditions of the Lemma imply that $a - c \notin Q$. In third case ($k = -1$), we have $c = (-N, m_1, s_1)$, $a - c \notin Q$ again. Consider the second case $n_1 = N$. Then we use the well-known relation: if $a \in \{x_k = n\}$ and $b \in \{x_k = 0\}$ then

$$P(a, b) \cap \{x_k = 0\} = bR, \ P(a, b) \cap \{x_k = n\} = a + bR. \tag{34}$$

It follows that $c \in a + bR$ hence, $c = a + kb$ for some nonzero integer k. However, $a + kb$ lies in Q if and only if $k = -1$. Thus $c = a - b$ ■

Lemma 6. *Let the vector a_1 and a_2 be different long maximal elements of Q and the planes $P(a_k, b)$, $(k = 1, 2)$ satisfy the conditions of Lemma 1. Then the invariants (see (10))*

$$\mathbf{Re} z(-a_k) z(a_k - b) z(b), \ (k = 1, 2) \tag{35}$$

uniquely determine $z(b)$ if $z(a_k - b)$, $z(a_k)$, $(k = 1, 2)$ are known and

$$\mathbf{Im}(z(a_1 - b) z(a_1) z(-(a_2 - b)) z(-a_2)) \neq 0. \tag{36}$$

Proof: To prove this Lemma, we need only to note that the last inequality shows that the determinant of the linear form (with respect to the $x(b)$, $y(b)$) in the Lemma is not zero ■

Now, take $a_1 = (N, M - 1, S)$, $a_2 = (N, M - 1, S - 2)$, $b = (0, -1, N)$. Without loss of generality it can be assumed that $S - 2 \neq kN$. Otherwise we consider $a_2 = (N, M - 1, S - 4)$. By Lemma 4 the planes $P(a_k, b)$, $(k = 1, 2)$ satisfy the conditions of Lemma 1 and we find $z(b)$ by Lemma 5. Now, we assume that $S - 1 \neq kN$ without loss of generality. Otherwise we consider $S - 3$ instead of $S - 1$. By Lemma 4 the plane $P(a, b)$, where $a = (N, M - 1, S - 1)$, $b = (0, -1, N)$, satisfies the conditions of Lemma 1. Therefore the invariant (10) determines the sign of $z(N, M - 1, S - 1)$, i.e., the sign of d_{S-1}. From knowledge the sign of d_S, we found the sign of d_{S-2p} by (32). In the same way, from knowledge of the sign of d_{S-1}, we find the sign of d_{S-2p-1}. Thus, we found (31).

Step 3: In this step we find

$$z(N, m, s) \ \forall m, s, \tag{37}$$

using the following Lemma.

Lemma 7. *Suppose $c \in Q$ has two different decompositions*

$$c = a_1 + b_1,\ \ c = a_2 + b_2, \quad a_1, b_1, a_2, b_2 \in Q_N$$

such that $z^2(a_k),\ z(a_k - b_k),\ k = \overline{1,2}$ are known and

$$\mathbf{Im}(z^2(a_1)z(a_1 - b_1)z^2(-(a_2))z(-(a_2 - b_2))) \neq 0. \tag{38}$$

If the planes $P(a_k, b_k),\ (k = 1, 2)$ satisfy the conditions of Lemma 2, then the invariants (see (11)) $\mathbf{Re}z^2(-a_k)z(a_k - b_k)z(a_k + b_k),\ (k = 1, 2)$ uniquely determine $z(c)$.

The proof is the same with the proof of the Lemma 5.

Now, we find the Fourier coefficients (37) by induction. They were found in step 1 and step 2 for $m = M, M-1, M-2$. Let us find $z(N, m, s)$ assuming that, we have already found the $z(N, q, s)$ for $q = M, M-1, ..., m+1$. Clearly, for any $s \in [S, -S]$ there are different pairs (s_1, p_1), (s_2, p_2) such that

$$s_k + p_k = s, \quad s_k, p_k, s_k - p_k \in [S, -S], \quad s_k - 2p_k \notin [-S, S],\ k = 1, 2$$

Then by Lemma 3 (see case 3) the planes $P(a_k, b_k)$, where $a_k = (N, m+1, s_k)$, $b_k = (0, -1, p_k)$, $k = 1, 2$, satisfy the conditions of Lemma 2. Moreover $z(a_k)$, $z(a_k - b_k)$ are known by assumption of induction. Therefore the application of Lemma 6 yields $z(N, m, s)$.

Replacing the role of first coordinate by the role of second coordinate and then the role of first coordinate by the role of third coordinate, we find $z(n, m, s)$ for all boundary points (n, m, s) of Q.

4 Finding the Fourier Coefficient Corresponding to Interior Point of Q

In first and second steps of this section we use the following Lemma.

Lemma 8. *Suppose the vector (n, m) is maximal element of Z^2, i.e., greatest common divisor of n and m is 1 (gcd(n,m)=1).*

(a) If $n \neq 0$, then the plane $P((0, -M, S), (n, m, 0))$ contains only the following vectors of Q: $q(n, m, 0),\ \pm(0, -M, S) + q(n, m, 0)$, where $q \in Z$.

(b) If $|n| > \frac{N}{2}$, then the plane $P((n, m, s), (0, pn, 1))$ contains only the following vectors of Q: $q(0, pn, 1),\ \pm(n, m, s) + q(0, pn, 1)$, where $q \in Z$ and $p = 1, 2$.

Proof: (a) Any element (n_1, m_1, s_1) of this plane satisfy

$S(nm_1 - n_1m) = s_1Mn$. If $(nm_1 - n_1m) = 0$, then $s_1 = 0$. If $(nm_1 - n_1m) \neq 0$, then $s_1 = \pm S$ since S is prime number which is greater than n and M (see (6)). Hence, (n_1, m_1, s_1) belongs either to $\{x_3 = 0\}$ or to $\{x_3 = \pm S\}$. To complete the proof, it suffices to apply (34).

(b) The elements (n_1, m_1, s_1) of this plane satisfy

$$n(pns_1 - m_1) = n_1(pns - m). \tag{39}$$

The relation $gcd(n, m) = 1$ implies that $pns - m \neq 0,\ gcd(n, pns - m) = 1$. Therefore, it follows from inequalities $|n| > \frac{N}{2}$, $|n_1| \leq N$ and equality (39) that either $n_1 = 0$ or $n_1 = \pm n$. This together with (34) gives us the proof ■

Step 1: In first step of this section, we find

$$z(n, m, 0),\ z(n, 0, s),\ z(0, m, s)\ \ \forall n, m, s. \tag{40}$$

Let us find $z(n, m, 0)$. Without loss of generality it can be assumed that $m \geq 0$ and $n > 0$. Let p be greatest integer satisfying $pn \leq N$, $pm \leq M$. Clearly $q(n, m, 0) \in Q$ if and only if $-p \leq q \leq p$. Moreover $(0, -M, S) + q(n, m, 0) \in Q$ if $0 \leq q \leq p$ and $(0, M, S) + q(n, m, 0) \in Q$ if $-p \leq q \leq 0$. It follows from Lemma 7 (see part a) that the invariant (8) for $a_q = (0, -M, S) + q(n, m, 0)$ and the plane $P((0, -M, S), (n, m, 0))$ has the form

$\mathbf{Re} z(-(a_q)) \sum\limits_l c_l z(l(n, m, 0)) z(a_q - l(n, m, 0))$, where $q = 1, 2, ..., p$.

Similarly, we have $\mathbf{Re} z(-(b_q)) \sum\limits_l d_l z(l(n, m, 0)) z(b_q - l(n, m, 0))$, where $b_q = (0, M, S) + q(n, m, 0)$; $q = -1, -2, ..., -p$ and c_l, d_l are known constants. Since the Fourier coefficients $z(a_q),\ z(a_q - l(n, m, 0)),\ z(b_q),\ z(b_q - l(n, m, 0))$ are known from Section 2, we have $2p$ linear form with respect to $2p$ unknown $x(q(n, m, 0))$ and $y(q(n, m, 0))$. As a result we find these unknowns if the corresponding determinant $D(2p)$ is not zero. Let us show that the determinant is not identically zero. Let $x(q(n, m, 0))$ be the q-th and $y(q(n, m, 0))$ be the $(p+q)$-th unknown of the system, where $q = 1, 2, ..., p$. Similarly let the q-th linear form ot the first system be the q-th linear form of the joint system and the q-th linear form ot the second system be the $(p+q)$-th linear form of the joint system . Then the determinant $D(2p)$ can be written in the form

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix},$$

where $A=(a_{q,l})$, $B=(b_{q,l})$, $C=(c_{q,l})$, $D=(d_{q,l})$, $q,l=1,2,...,p$;

$$a_{q,l}=x(a_q)(c_l x(a_{q-l})+c_{-l}x(a_{q+l}))+y(a_q)(c_l y(a_{q-l})+c_{-l}y(a_{q+l})),$$

$$b_{q,l}=x(a_q)(c_l y(a_{q-l})-c_{-l}y(a_{q+l}))+y(a_q)(c_l x(a_{q-l})-c_{-l}x(a_{q+l})),$$

$$c_{q,l}=x(b_{-q})(d_l x(b_{-q-l})+d_{-l}x(b_{-q+l}))+y(b_{-q})(d_l y(b_{-q-l})+d_{-l}y(b_{-q+l})),$$

$$d_{q,l}=x(b_{-q})(d_l y(b_{-q-l})-d_{-l}y(b_{-q+l}))+y(b_{-q})(d_l x(b_{-q-l})-d_{-l}x(b_{-q+l})).$$

The q-th and $(p+q)$-th diogonal elements ($a_{q,q}$ and $d_{q,q}$) of the determinant contain the summand $x(a_q)c_q x(a_0)$ and $x(b_{-q})d_{-q}y(b_0)$ respectively. The non-diogonal elements do not contain neither $x(a_0)$ nor $y(b_0)$. Therefore the determinant contain the summand $\sqcap_{q=1,2,\ldots,p}(x(a_q)c_q x(a_0)x(b_{-q})d_{-q}y(b_0))$ which can not be canseled by the other summand of the determinant. Note that this determinant is a homogeneous polynomial of $x(a_q),y(a_q),x(b_{-q}),y(b_{-q})$, for $q=1,2,...,p$, of degree $4p$. Therefore the zero set of the determinant is either zero or a union of conical surfaces.

In the same way we find the other term of (40).

Remark 1: Taking into account that the cases $n=0,\pm N;\ \ m=0,\pm M;\ \ s=0,\pm S$ have been considered and $z(-n,-m,-s)$ is the complex conjugate of $z(n,m,s)$, further we assume that $n\neq 0,\pm N;\ m\neq 0,\pm M;\ \ 0<s<S.$

Step 2: In this step we find $z(n,m,s)$ for all (n,m,s) satisfying

$$gcd(n,m)=1,\quad |n|>\frac{N}{2}. \tag{41}$$

Since the $z(n,m,s)$ is known for $s=S$ (see Section 2), we find $z(n,m,s)$ for all (n,m,s) satisfying (41) by induction. Assume that these Fourier coefficients are already known for $s>k>0$. If

$$z(q(0,n,1))=0,\quad \text{for } |n|>\frac{N}{2} \text{ and } \quad |q|>1, \tag{42}$$

then by Lemma 7 (see part (b)) the invariant (8) for $a=(n,n+m,k+1)$ and the plane $P((n,m,k),(0,n,1))$ has the form

$$\mathbf{Re}z(-a)(z(n,m,k)z(0,n,1)+cz(n,m+2n,k+2)z(0,-n,-1)).$$

Similarly, the invariant (8) for $a=(n,2n+m,k+1)$ and the plane

$P((n,m,k),(0,2n,1))$ has the form

$$\mathbf{Re}z(-a)(z(n,m,k)z(0,2n,1)+dz(n,m+4n,k+2)z(0,-2n,-1)),$$

where c, d are known constants. Here all Fourier coefficients are known except $z(n,m,k)$, since its either were found in step 1 or are known by induction assumption. Moreover if m and n have different signs then the condition (6) shows that $-M < m+n,\ m+2n < M$ and the coefficient of $z(n,m,k)$ is not zero. If they have same signs, then we replace the vector $(0,n,1)$ by $(0,-n,1)$ and apply the preceding arguments. So from these invariants we find $z(n,m,k)$ under the condition that the corresponding determinant is not zero.

Remark 2: In this step, we only used that maximal first coordinate N of vectors of Q is less than half of maximal second coordinate M. Since the maximal first coordinate N is less than half of maximal third coordinate S and the maximal second coordinate M is less than half of maximal third coordinate S arguing as above, we find $z(n,m,s)$ for all (n,m,s) satisfying

$$gcd(n,s)=1,\ |n|>\frac{N}{2} \text{ or } gcd(m,s)=1,\ |m|>\frac{M}{2}.$$

Here instead the condition (42) we have to use the conditions:

$$z(q(0,1,n))=0,\ z(q(1,0,m))=0 \text{ for } |n|>\frac{N}{2},\ \ |q|>1,\ |m|>\frac{M}{2}.$$

Step 3: In this step, we find $z(n,m,s)$ satisfying

$$\gcd(n,m)=1,\quad |n|<\frac{N}{2},\ \left|\frac{m}{n}\right|<\frac{M+1}{N+1}. \tag{43}$$

For this we need to analyze some property of maximal elements of Z^2. Since $nq-pm=\langle(n,m),(q,-p)\rangle$ from the definition of maximal elements of Z^2 we obtain that (n,m) is maximal if and only if for each $l\in Z$ there is $(p,q)\in Z^2$ such that $nq-pm=l$. Denote $A_l(n,m)=\{(p,q)\in Z^2: nq-pm=l\}$,

$Q(3)=\{(n,m): |n|\le N,\ |m|\le M\}$.

Definition 9. *A maximal vector (n,m) is said to be long maximal (with respect to $Q(3)$) if $(n,m)\in Q(3)$, but $2(n,m)\notin Q(3)$, i.e., if $(n,m,0)$ is long maximal vector of Q. In other words a vector (n,m) of $Q(3)$ is long maximal if $gcd(n,m)=1$ and at least one of the following inequalities holds $|n|>\frac{N}{2}$, $|m|>\frac{M}{2}$.*

Lemma 10. *Let (n', m') and (n'', m'') be longest element of $A_1(n,m)\bigcap Q(3)$ and $A_{-1}(n,m)\bigcap Q(3)$ forming an acute angle with (n,m), where (n,m) is maximal.*

(a) The vector $(n', m') + p(n,m)$ is maximal for all $p \in Z$.

(b) If p is positive integer, then $(n', m') + p(n,m) \notin Q(3)$, i.e., at least one of the following inequalities holds $|n' + pn| > N$,$|m' + pm| > M$.

(c) If (n,m) satisfies (43), then $|n' + n| > N$ hence, $|n'| > \frac{N}{2}$.

(d) If $|n| < \frac{N}{2}$ and $|m| < \frac{M}{2}$, then (n', m') is long maximal. All statements (a)-(d) are hold, if we replace n' and m' by n'' and m''.

Proof: (a) By definition of (n', m') we have

$$n(m' + pm) - m(n' + pn) = nm' - mn' = 1. \tag{44}$$

If the vector $(n', m') + p(n,m)$ is not maximal for some p, then $n' + pn = lk$ and $m' + pm = lq$ for some $l > 1$, where $k, q \in Z$. Therefore it gives a contradiction in (44) as $l(nq - mk) = 1$.

(b) Suppose to the contrary that there is a positive integer p such that $(n', m') + p(n,m) \in Q(3)$. Then $(n', m') + p(n,m)$ belongs to $A_1(n,m)$, (see (44)) longer than (n', m') and forms acute angle with (n,m). However it contradicts to the definition (n', m').

(c) If the inequality $|n' + n| > N$ is not true, then part (b) of this Lemma gives $|n' + n| \leq N$ and $|m' + m| > M$. Using these inequalities and (44) for $p = 1$, we obtain

$$\left|\frac{n}{m}\right| = \left|\frac{n + n'}{m + m'} + \frac{1}{m(m + m')}\right| \leq \frac{N + 1}{M + 1},$$

which contradicts to (43). Thus $|n' + n| > N$.

(d) follows from (b) by taking $p = 1$. Similarly we prove these statements for n'' and m'' ■

Now let us find $z(n,m,s)$ satisfying (43). The invariant (8) for $a' = (n', m', s - S)$ and the plane $P((n,m,s), a')$ has form

$$\mathbf{Re} z(-a')(z(n,m,s)z(n' - n, m' - m, -S) + \sum_{a_1} c(a_1)z(a_1)z(a' - a_1)). \tag{45}$$

Similarly (8) for $a'' = (n'', m'', s - S)$ and the plane $P((n,m,s), a'')$ has form

$$\mathbf{Re} z(-a'')(z(n,m,s)z(n'' - n, m'' - m, -S) + \sum_{a_2} c(a_2)z(a_2)z(a'' - a_2)), \tag{46}$$

where $a_i = (n_i, m_i, s_i)$ and $c(n_i, m_i, s_i)$, $(i = 1, 2)$ are known constants. Here the sums are taken under the conditions that

$$(n_1, m_1, s_1), (n' - n_1, m' - m_1, s - S - s_1) \in P((n, m, s), (n', m', s - S)) \bigcap Q,$$

$$(n_2, m_2, s_2), (n'' - n_2, m'' - m_2, s - S - s_2) \in P((n, m, s), (n'', m'', s - S)) \bigcap Q,$$

and

$$(n_1, m_1, s_1) \neq 0, (n, m, s), (n', m', s - S), (n' - n, m' - m, -S), \tag{47}$$

$$(n_2, m_2, s_2) \neq 0, (n, m, s), (n'', m'', s - S)(n'' - n, m'' - m, -S). \tag{48}$$

Let us show that all terms under the sign of the sum in (45) is zero. Since (n, m), (n', m') are maximal and $nm' - mn' = 1$, they form a basis in Z^2. Therefore

$$(n_1, m_1) = l(n', m') + p(n, m), \tag{49}$$

for some integers l and p. Then the condition

$(n_1, m_1, s_1) \in P((n, m, s), (n', m', s - S))$ is equivalent to

$$s_1 = (s - S)l + sp. \tag{50}$$

Without loss of generality, it can be assumed that $l \geq 0$. There are the cases: $l = 1$, $l > 1$, $l = 0$.

Case 1. $(l = 1)$ In this case $(n_1, m_1) = (n', m') + p(n, m)$. By part (b) of Lemma 8 we need to consider $p \leq 0$. If $p = 0$, then by (49) and (50), we have $s_1 = s - S$ and $(n_1, m_1, s_1) = (n', m', s - S)$ which contradicts to (47). If $p = -1$, then again by (49) and (50), we have $(n_1, m_1, s_1) = (n' - n, m' - m, -S)$ which again contradicts to (47). If $p < -1$, then by (50) and Remark 1 we have $s_1 = -S + (1 + p)s < -S$ and $z(n_1, m_1, s_1) = 0$.

Case 2. ($l > 1$) By (49), we have

$$n_1 = ln' + pn = l(n' + n) + (p - l)n. \tag{51}$$

By part (c) of Lemma 8, $|n + n'| > N$. We shall consider the case $n + n' > N$ (The case $n + n' < -N$ is similar). Then both n and n' are positive. Hence by (43) and Remark 1 we have the inequalities

$$n + n' > N; \quad 0 < 2n < N; \quad -N \leq n_1 \leq N; \quad l > 1; \quad 0 < s < S \tag{52}$$

Using these inequalities from (51), we get

$$N \geq l(n'+n)+(p-l)n > lN+(p-l)n,\ (l-p)n>(l-1)N,\ l-p>2l-2,$$

$$l+p\leq 1,\ s(l+p)<S \text{ and } -Sl+s(l+p)<-S.$$

Therefore from (50) we obtain $s_1=-Sl+(p+l)s<-S$, which means that $z(n_1,m_1,s_1)=0$.

Case 3. $(l=0)$ We replace the role of (n_1,m_1,s_1) by the role $(n'-n_1,m'-m_1,s-S-s_1)$ and repeat the case $l=1$.

Thus the terms under the sum in (45) and similarly in (46) are zero. On the other hand, the Fourier coefficients $z(n'-n,m'-m,-S)$, $z(n''-n,m''-m,-S)$ are known from Section 3 and $z(n',m',s-S)$, $z(n'',m'',s-S)$ are known from step 2 of this section (see parts (a), (c) of Lemma 8). Therefore, the invariants (45) and (46) give us $z(n,m,s)$ (see Lemma 5).

Remark 3: Using Remark 2, replacing the role of second coordinate by the role of third coordinate and then the role of first coordinate by the role of second coordinate, and arguing as above we find $z(n,m,s)$ satisfying

$$gcd(n,s)=1,\ |n|<\frac{N}{2},\quad \left|\frac{s}{n}\right|<\frac{S+1}{N+1}$$

or $gcd(m,s)=1,\quad |m|<\frac{M}{2},\quad \left|\frac{s}{m}\right|<\frac{S+1}{M+1}$.

Step 4: In this step we find all $z(qn,qm,s)$, except $z(nq_0,mq_0,s)$ for $|s|>\frac{2}{3}S$, satisfying

$$gcd(n,m)=1,\quad |n|<\frac{N}{2},\quad \left|\frac{m}{n}\right|<\frac{M-N}{N+1}, \tag{53}$$

where $q=1,2,...,q_0$ and the integer q_0 is taken so that $q_0(n,m)\in Q(3)$, but $(1+q_0)(n,m)\notin Q(3)$. For $q=1$ this Fourier coefficient is known from step 3, since the condition (53) implies (43). Therefore we take $2\leq q\leq q_0$. By part (c) of Lemma 8 $|n+n'|>N$. We shall consider the case $n+n'>N$. (The case $n+n'<-N$ is similar) Then both n and n' are positive. The relation $q_0(n,m)\in Q(3)$ implies that $q_0 n\leq N$ and by (53), we have the inequalities

$$n+n'>N;\quad 0<2n<N;\quad 2\leq q\leq q_0;\quad q_0 n\leq N. \tag{54}$$

Now assuming that $z(kn,km,s)$ is known for $k<q$, we find $z(qn,qm,s)$. For this we consider the invariant (8) for the vector $a'=(n',m',s-S)$ and the plane $P((qn,qm,s),a')$. It has form

$$\mathbf{Re}(z(-a')(z(qn,qm,s)z(a'-(qn,qm,s))+\sum_{a_1}c(a_1)z(a_1)z(a'-a_1)). \quad (55)$$

Similarly (8) for $a''=(n'',m'',s-S)$ and the plane $P((qn,qm,s),a'')$ has form

$$\mathbf{Re}(z(-a'')(z(qn,qm,s)z(a''-(qn,qm,s))+\sum_{a_2}c(a_2)z(a_2)z(a''-a_2)), \quad (56)$$

where $a_i=(n_i,m_i,s_i)$ and $c(n_i,m_i,s_i)$, $(i=1,2)$ are known constants. Here the sums are taken under the conditions that

$$(n_1,m_1,s_1),(n'-n_1,m'-m_1,s-S-s_1)\in P((qn,qm,s),(n',m',s-S))\bigcap Q,$$

$$(n_2,m_2,s_2),(n''-n_2,m''-m_2,s-S-s_2)\in P((qn,qm,s),(n'',m'',s-S))\bigcap Q,$$

and

$$(n_1,m_1,s_1)\neq 0,((qn,qm,s),(n',m',s-S),(n'-qn,m'-qm,-S), \quad (57)$$

$$(n_2,m_2,s_2)\neq 0,(qn,qm,s),(n'',m'',s-S)(n''-qn,m''-qm,-S). \quad (58)$$

Let us prove that all terms under the sign of sum in (55) either is zero or is known. This similar as in step 3. Using (49) we can easily verify that the condition $(n_1,m_1,s_1)\in P((qn,qm,s),(n',m',s-S))$ is equivalent to

$$qs_1=q(s-S)l+sp. \quad (59)$$

Without loss of generality, it can be assumed that $l\geq 0$. The investigation of the case $l=0$ can be reduced to the investigation of the case $l=1$ (see case 3 in step 3). So we need consider the cases $l=1$, $l>2$ and $l=2$.

Case 1. $(l=1)$ In this case $(n_1,m_1)=(n',m')+p(n,m)$. By part (b) of Lemma 8 we need to consider $p\leq 0$. If $p<-q$, then we have $s_1<-S$ by (59). If $p=-q$, then $s_1=-S$ and $(n_1,m_1,s_1)=(n'-qn,m'-qm,-S)$

which contradicts to (57). The case $p = 0$ also contradicts to (57). So we consider the case $0 > p > -q$. By (44), we have

$$\frac{m' + pm}{n' + pn} = \frac{m}{n} + \frac{1}{n(n' + pn)}. \tag{60}$$

Here $n \neq 0$ by Remark 1. Combining the inequality $-p < q$ with the inequalities (54) we get $n' + pn = n' + n - (-p+1)n > N - qn \geq N - q_0 n \geq 0$. Now from (60) and (53), we get

$$\left| \frac{m' + pm}{n' + pn} \right| < \frac{M + 1}{N + 1}.$$

So if $|n' + pn| < \frac{N}{2}$, then $z(n_1, m_1, s_1) \equiv z(n' + pn, m' + pm, s)$ is known from step 3 (see part (a) of Lemma 8). If $|n' + pn| > \frac{N}{2}$, then this Fourier coefficient is known from step 2. Since $-p < q$ the Fourier coefficient

$z(n' - n_1, m' - m_1, s - S - s_1) \equiv z(pn, pm, s - S - s_1)$

is known by assumption of induction.

Case 2. $(l > 2)$ By (49) we have $-pn = ln' - n_1$. Taking into account that $n_1 \leq N$ and using (54) we get $-pn \geq ln' - N > l(N - n) - N = N(l-1) - ln$ and $-p \geq (l-1)q_0 - l + 1 = (l-1)(q_0 - 1)$. This together with (59) implies

$$-s_1 \geq Sl - sl + \frac{s(l-1)(q_0 - 1)}{q}. \tag{61}$$

Since $(l - 1) \geq 2$, $q_0 \geq 2$ and $0 < q \leq q_0$, we have $(l - 1)(q_0 - 1) \geq q$. From (61), taking into account that $0 < s < S$ (see Remark 1), we obtain $-s_1 > S$ and $z(n_1, m_1, s_1) = 0$.

Case 3. $(l = 2)$ In this case (61) has form $-s_1 \geq 2S + (\frac{q_0 - 1}{q} - 2)s$. If $q \leq q_0 - 1$, then $-s_1 > S$ due to $s < S$. Similarly if $q = q_0$ and $s < \frac{2}{3}S$ then again $s_1 > S$, i.e., in these cases $z(n_1, m_1, s_1) = 0$.

Thus we proved that in all cases except $q = q_0$ and $s > \frac{2}{3}S$, all terms under the sum in (55) (similarly in(56)) either is zero or is known. Moreover $z(-(n', m', s - S))$, $z(-(n'', m'', s - S))$ are known from step 2 of this section (see parts (a), (c) of Lemma 8) and $z(n'' - qn, m'' - qm, -S)$, $z(n' - qn, m' - qm, -S)$ are known from Section 3. By Lemma 5 the invariants (55), (56) give us $z(qn, qm, s)$ for all cases except $q = q_0$ and $s > \frac{2}{3}S$.

Remark 4: Using Remark 3, replacing the role of second coordinate by the role of third coordinate and arguing as above we find all $z(qn, m, qs)$, except $z(nq_1, m, sq_1)$ for $|m| > \frac{2}{3}M$, satisfying

$$gcd(n,s)=1, \quad |n| < \frac{N}{2}, \quad \left|\frac{s}{n}\right| < \frac{S-N}{N+1},$$

where $q = 1, 2, ..., q_1$ and the integer q_1 is taken so that $q_1(n,s) \in Q(2) \equiv \{(n,s) : |n| \leq N, \ |s| \leq S\}$, but $(1+q_1)(n,s) \notin Q(2)$.

Similarly replacing the role of first coordinate by the role of second coordinate we find all $z(n, qm, qs)$, except $z(n, mq_2, sq_2)$ for $|n| > \frac{2}{3}N$, satisfying

$$gcd(m,s)=1, \quad |m| < \frac{M}{2}, \quad \left|\frac{s}{m}\right| < \frac{S-M}{M+1},$$

where $q = 1, 2, ..., q_2$ and the integer q_2 is taken so that $q_2(m,s) \in Q(1) \equiv \{(m,s) : |m| \leq M, \ |s| \leq S\}$, but $(1+q_2)(m,s) \notin Q(1)$. Since $S - M > 2(M+1)$ (see (6)) we have found $z(n,m,s)$ satisfying $|s| \leq |2m|$ and $|n| < \frac{2N}{3}$.

Step 5: In this step we find $z(n,m,s)$ satisfying $|m| > \frac{M}{2}, \ gcd(n,m) = 1$.

The case $|n| > \frac{N}{2}$ is considered in step 2.Therefore we consider the case $|n| < \frac{N}{2}$.By Remark 4, these Fourier coefficients are known for $|s| \leq 2|m|$. So we study the case $s > 2|m|$ (The case $s < -2|m|$ is similar). We use the following lemma.

Lemma 11. *If (n,m) is maximal then for each integer s there are integers s' and s'' from interval $[s-|m|, s)$ such that $ns'-s = mp'+1, \ \ ns''-s = mp''-1$, where p', p'' are integers.*

Proof: Since (n,m) is maximal, there are integers p and q such that $nq-mp = s+1$. Moreover for every integer k we have $n(q+km)-m(p+kn) = s+1$. Therefore there is an integer k' such that $q+k'm$ lies in $[s-|m|, s)$ and $n(q+k'm) - m(p+k'n) = s+1$. Now taking $p' = p+nk'$, $s' = q+mk'$ we obtain the first equality of this Lemma. To obtain the second equality, we take $s-1$ instead of $s+1$ and repeat this argument ■

Thus in this step we have the inequalities

$$s > 2|m| > M; \quad |m| < s - |m| \leq s', \quad s'' < s < S. \tag{62}$$

Now consider the plane $P((n,m,s),(1,0,s'))$, where n, m, s' are defined in Lemma 9. The point (n_1, m_1, s_1) belongs to this plane if and only if

$$m(n_1 s' - s_1) = m_1(s - ns'). \tag{63}$$

If $m_1 = 0$, then $n_1 s' = s_1$ and $(n_1, m_1, s_1) = n_1(1, 0, s')$. Consider the case $m_1 \neq 0$. It follows from (62) and Lemma 9 that $s - ns' \neq 0$ and $gcd(s - ns', m) = 1$. Therefore taking into account that $|m_1| \leq M < 2|m|$, from (63) we get that $m_1 = 0$ or $m_1 = \pm m$. By (34) the intersection of this plane with Q consists of the points

$q(1, 0, s')$, $\pm(n, m, s) + q(1, 0, s')$. We assumed that (see Remark 2)

$$z(q(1, 0, s')) = 0 \text{ for } |s'| > M/2 \text{ and } |q| > 1. \tag{64}$$

Then the invariant (8) for (n, m, s) and the plane $P((n, m, s), (1, 0, s'))$ has form

$\mathbf{Re} z(-(n, m, s))(z(n-1, m, s-s')z(1, 0, s') + cz(n+1, m, s+s')z(-1, 0, -s'))$.

Similarly we have

$\mathbf{Re} z(-(n, m, s))(z(n-1, m, s-s'')z(1, 0, s'') + dz(n+1, m, s+s'')z(-1, 0, -s''))$.

Since $|s - s'|$, $|s - s''| \leq |m|$ (see Lemma 9),and $|n - 1| < \frac{2N}{3}$ the Fourier coefficients $z(n - 1, m, s - s')$, $z(n - 1, m, s - s'')$ are known (see the end of Remark 4). Moreover $z(1, 0, s')$, $z(1, 0, s'')$ are known from step 1. If $s = S - 1$, then $s' + s$, $s + s'' > S$ (see(62)) and $z(n + 1, m, s + s')$, $z(n + 1, m, s + s'') = 0$. Hence in case $s = S - 1$ all Fourier coefficients in above invariants except $z(n, m, s)$ are known. Therefore it follows from Lemma 5 that these invariants determine $z(n, m, S - 1)$. Now we find $z(n, m, s)$ by induction method. Indeed, if we assume that $z(n, m, S-2)$, $z(n, m, S-3)$,..., $z(n, m, s + 1)$ are known, then we see that all Fourier coefficients in these invariants, except $z(n, m, s)$, are known.

Remark 5: The combination of step 2 and step 5 gives: The Fourier coefficient $z(n, m, s)$ is known if (n, m) is long maximal.

Step 6: In this step we find all Fourier coefficients. The vectors of Q can be written in the form (qn, qm, s), where $gcd(n, m) = 1$; $q = 1, 2, ..., q_0$ and the integer q_0 is taken so that $q_0(n, m) \in Q(3)$, but $(1 + q_0)(n, m) \notin Q(3)$. If $q_0 = 1$, then (n, m) is long maximal and $z(n, m, s)$ is known by Remark 5. Therefore we take $q_0 \geq 2$. Let us find $z(qn, qm, s)$ by induction. For $q = 1$ we find these Fourier coefficients repeating step 3. The step 3 is based on step 2. In step 2 $z(n, m, s)$ is found for all long maximal (n, m) satisfying $|n| > N/2$. Now we repeat step 3 using Remark 5, where $z(n, m, s)$ is found for all long maximal (n, m) without condition $|n| > N/2$. So we again consider the invariants (45), (46). By Lemma 8 the vectors (n', m'), (n'', m'') are long maximal and $z(n', m', s - S)$, $z(n'', m'', s - S)$ are known (see Remark 5). The Fourier coefficients $z(n' - n, m' - m, -S)$, $z(n'' - n, m'' - m, -S)$ are known from Section 3. To find $z(n,m,s)$ from (45), (46) it suffices to prove that all terms under the sign of the sums in (45), (46) are zeros. For this we repeat case 1 and case 3 word by word. Consider case 2. If $|n+n'|>N$, then we again repeat it word by word. If $|n+n'|\leq N$, then we have $|m+m'|>N$ by part (b) of Lemma 8. Now we replace the roles of first coordinates $n,\ n',\ N$ by the roles of second coordinates $m,\ m',\ M$. More precisely, instead the inequalities (52) we use

$$m+m'>M;\quad 0<2m<M;\quad -M\leq m_1\leq M;\quad l>1;\ 0<s<S \tag{65}$$

and arguing as in case 2 we obtain that $z(n_1,m_1,s_1)=0$. At the end we find the Fourier coefficient $z(n,m,s)$ if $gcd(n,m)=1$ (without inequalities in (43)). Using it and repeating step 4, we find $z(nq,mq,s)$ satisfying (53) without inequalities in it (exsept $z(nq_0,mq_0,s)$ for $|s|>\frac{2}{3}S$). We only note the following concerning the repetition of step 4. In case 1, we have $(n_1,m_1)=(n',m')+p(n,m)$. By part (a) of Lemma 8 (n_1,m_1) is maximal and $z(n_1,m_1,s_1)$ is known. The repetition of remained part of step 4 is similar to the repetition of step 3. So we have found all Fourier coefficients, exsept $z(nq_0,mq_0,s)$ for $S>|s|>\frac{2}{3}S$. To find its ,i.e., $z(a)$ for

$$a\in Q_1\equiv\{(nq_0,mq_0,s):S>|s|>\frac{2}{3}S,gcd(n,m)=1\}$$

we consider the invariants (12) in the plane $P((n,m,0),(0,0,1))$ for the directional potential $q^{\delta}(x)$, where δ lies on line $(nq_0,mq_0,l)R$. It follows from (34) that this plane contains only the vectors $(0,0,s)$ and $\pm(nq,mq,s)$ for $s,q\in Z$. Therefore these invariants have the form

$$\mathbf{Re}z(-(nq_0,mq_0,l))(\sum_{s_1}z(nq_0,mq_0,s_1)z(0,0,l-s_1)c(s_1)$$

$$+\sum_{(n_1,m_1,s_1)}z(n_1,m_1,s_1)z(nq_0-n_1,mq_0-m_1,l-s_1)c(n_1,m_1,s_1)),$$

where $c(s_1),\ d(s_1),\ c(n_1,m_1,s_1)$ are known constants. Here $z(n_1,m_1,s_1)$ and $z(nq_0-n_1,mq_0-m_1,s-s_1)$ are known since (n_1,m_1,s_1) and $(nq_0-n_1,mq_0-m_1,s-s_1)$ belong to the plane $P((n,m,0),(0,0,1))$ and clearly, have the form (qn,qm,k), where $0<q<q_0$. We enumerate these invariants accoding to the change of l as follows:

$$l=1,2,...,[\frac{1}{3}S],-1,-2,...,-[\frac{1}{3}S],[\frac{2}{3}S]-[\frac{1}{3}S]+1,[\frac{2}{3}S]-[\frac{1}{3}S]+2,[\frac{2}{3}S],$$

$$-([\frac{2}{3}S]-[\frac{1}{3}S]+1), -([\frac{2}{3}S]-[\frac{1}{3}S]+2), -[\frac{2}{3}S],$$

where $[a]$ is the integer part of a. After that we enumerate the unknowns in the following order:

$$x(nq_0, mq_0, [\frac{2}{3}S]+1), x(nq_0, mq_0, [\frac{2}{3}S]+2), ..., x(nq_0, mq_0, S-1),$$

$$x(nq_0, mq_0, -([\frac{2}{3}S]+1)), x(nq_0, mq_0, -([\frac{2}{3}S]+2)), ..., x(nq_0, mq_0, -(S-1)),$$

$$y(nq_0, mq_0, [\frac{2}{3}S]+1), y(nq_0, mq_0, [\frac{2}{3}S]+2), ..., y(nq_0, mq_0, S-1),$$

$$x(nq_0, mq_0, -([\frac{2}{3}S]+1)), x(nq_0, mq_0, -([\frac{2}{3}S]+2)), ..., x(nq_0, mq_0, -(S-1)),$$

The obvious relation $[\frac{2}{3}S]+[\frac{1}{3}S]=S-1$ shows that we have $4[\frac{1}{3}S]$ linear forms with $4[\frac{1}{3}S]$ unknowns $x(nq_0, mq_0, s),\ y(nq_0, mq_0, s)$, where $\frac{2}{3}S < s < S$. Thus we uniquely determine these unknowns under the condition that the corresponding determinant $D(4[\frac{1}{3}S])$ is not zero. Let us show that the determinant is not identically zero. The multiplication of diogonal elements of the determinant contain a summand with the factor

$$(x(0,0,[\frac{2}{3}S]))^{2[\frac{S}{3}]})(x(0,0,[\frac{1}{3}S]))^{2[\frac{S}{3}]})$$

This summand can not be canseled by the other summand of the determinant,since the nondiogonal elements do not contain neither $x(0,0,[\frac{2}{3}S])$ nor $x(0,0,[\frac{1}{3}S])$. Note that this determinant is a homogeneous polynomial, hence its zero set is either zero or a union of conical surfaces.

5 Main Results

In the previous Sections, we considered the set of polynomials with support $\{a \in \Gamma : z(a) \neq 0\} = Q_{N,M,S} \backslash Q_2$, (see (5), conditions (42), and the similar conditions in the end of remark 2), where

$$Q_2 = \{q(0,n,1), q(0,1,n), q(1,0,m) : |q| > 1, |n| > \frac{N}{2}, |m| > \frac{M}{2}\}.$$

Moreover we had the conditions (23), (30) and $f_3(V_j) - f_3(V_k) \neq 0$, for $j \neq k$ (see (32)), on the argument of the Fourier coefficients.Then we had conditions (36) and (38). In Section 4 we assumed that some determinants $D(j)$ of order j are not zero. If $j = 2$ then these assumptions are just (36). In Step 1 and Step 6 of that section we had the conditions $D(2p) \neq 0$ and $D(4[\frac{1}{3}S]) \neq 0$. All the listed conditions, exsept $D(4[\frac{1}{3}S]) \neq 0$,(We call these conditions as Condition A) are related to the $x(a)$, $y(a)$ for $\|a\| > \frac{N}{2}$. It means that if we consider the set $E(N,M,S)$ of the polynomials of the form

$$q(x) = \sum_{a \in G(N,M,S)} z(a) e^{i\langle a,x \rangle}, \tag{66}$$

where $G(N,M,S) = Q_{N,M,S} \setminus (Q_1 \cup Q_2)$ (see Step 6) then for $\|a\| < \frac{N}{2}$ we have only the condition $z(a) \neq 0$. Since the set $G(N,M,S)$ contains the ball $\{\| \ (n,m,s) \ \|< \frac{N}{2}\}$, the polynomials (66) can be used for appoximation of smooth functions. Denote by $E_0(N,M,S)$ the set of all elements of $E(N,M,S)$ satisfying Condition A.

Now consider a triple sequence $\{(N_k, M_k, S_k)\}$ such that for all k the triple (N_k, M_k, S_k) satisfies the conditions which are satisfied for (N,M,S) (see (6)). So $N_k, \ M_k, \ S_k$ are prime numbers satisfying

$$M_k > 2N_k, \ S_k > 3M_k + 2, \ N_{k+1} > 2N_k, \ M_{k+1} > 2M_k, \ S_{k+1} > 2S_k, \ N_1 \gg 1.$$

Denote by $E(N_k, M_k, S_k)$ and $E_0(N_k, M_k, S_k)$ the set obtained from $E(N,M,S)$ and $E_0(N,M,S)$ by substitution (N_k, M_k, S_k) for (N,M,S). Let

$$E_0 = \bigcup_{k=1}^{\infty} E_0(N_k, M_k, S_k).$$

Obviously the set E_0 is dense in $L_2(F)$.

Theorem 12. *Suppose that $q(x) \in E_0$. If the invariants (1), (2), (4) are known, then one can determine constructively and uniquely (in E_0 modulo translations $q(x) \to q(-x),\ q(x) \to q(x+\tau)$) the potential $q(x)$.*

Proof: The vector $(N_k, 1, 0)$ is maximal for each $N_k > 1$, i.e., the invariants $\|q^{(N_k,1,0)}(x)\|$ (see (1)) are given. By definition of the set E_0, the number

$$k \equiv \{\max k : \quad \text{for which } \|q^{(N_k,1,0)}(x)\| \neq 0\}$$

is finite. Therefore, $q(x) \in E_0(N_k, M_k, S_k)$. The statement of this Theorem is proved for this set in Sections 3, 4 ■

Now consider the set $C^p(F)$ of p times differentiable periodic function, where $p \gg 1$. Assign to every function $q(x)$ of $L_2(F)$ the vector $V(q) = (x(a_1), y(a_1), x(a_2), y(a_2), ...,)$ of l_2, where $x(a_j)$ and $y(a_j)$ are real and imaginary part of the Fourier coefficient $z(a_j)$ of $q(x)$. Here the vectors $a_1,\ a_2, ...$ of the lattice Γ are numerated in non-decreasing order of $\|a_j\|$. Since $z(-a_j)$ is the complex conjugate of $z(a_j)$, we include one of the vector $a_j, -a_j$ to the list $a_1,\ a_2, ...$ Hence we defined the isometric mapping V from $L_2(F)$ onto l_2 taking $q(x)$ to $V(q)$. For fixing the translations $q(x) \to q(-x),\ q(x) \to q(x+\tau)$ we set

$$\alpha(a_i) = 0,\ \alpha(a_4) \in (0, \pi),\ \textit{or}\ \ y(a_i) = 0, y(a_4) > 0 \tag{67}$$

where $i = 1, 2, 3,\ a_1 = (1, 0, 0),\ a_2 = (0, 1, 0),\ a_3 = (0, 0, 1),\ a_4 = (1, 1, 0)$. Denote by $\widetilde{C}^p(F)$, $\widetilde{E}(N_k, M_k, S_k)$ and $\widetilde{E}_0(N_k, M_k, S_k)$ the set of all function of $C^p(F)$, $E(N_k, M_k, S_k)$ and $E_0(N_k, M_k, S_k)$ satisfying (67). In other words $\widetilde{A}$ denotes the set A with fixed translations $q(x) \to q(-x),\ q(x) \to q(x+\tau)$. Let

$$\widetilde{l}_2 = V(\widetilde{C}^p(F)),\ \widetilde{l}_2(p_k) = V(\widetilde{E}(N_k, M_k, S_k)),\ \widetilde{l}_2(p_k, 0) = V(\widetilde{E}_0(N_k, M_k, S_k)),$$

where p_k is the number of elements of $G(N_k, M_k, S_k)$. The sets $\widetilde{l}_2(p_k, 0)$ and $\widetilde{l}_2(p_k)$ are $(p_k - 3)$-dimensional (see (67)) subsets of l_2. The Condition A shows that the set $\widetilde{l}_2(p_k, 0)$ is obtained from $\widetilde{l}_2(p_k)$ by eliminating a finite number of conical surfaces $A_1,\ A_2,...,\ A_q$ of dimension less than $p_k - 3$.

In [5] (see introduction of [5]) by given Floquet spectrum we constructively determined the spectral invariants

$J_k(a, \beta, q) = \int_F |q_{a,\beta}(x)|^2 A_k(x)dx, k = 0, 1, 2, ...,$ where $A_k(x)$ is expressed via the $q^a(x) = Q(s)$ ($s = \langle x, a\rangle$) and is defined from the decomposition

$\left|\varphi^a_{n,v}(s)\right|^2 = A_0 + \frac{A_1(s)}{n} + \frac{A_2(s)}{n^2} + ...$
Here $\varphi^a_{n,v}(s)$ is eigenfunction of the boundary value problem
$- \mid a \mid^2 y''(s) + Q(s)y(s) = \mu y(s),\ y(s+2\pi) = e^{i2\pi v}y(s).$

In case (3) we have $A_0(x) = 1, A_1(x) = 0,\ A_2 = \frac{q^a(x)}{2} + c_1 \left|q_a\right|^2,\ A_3 = c_2 q^a(x) + c_3 \left|q_a\right|^2,\ A_4(x) = c_4 q^a(x) + c_5(q_a^2 e^{i2\langle a,x\rangle} + q_{-a}^2 e^{-i2\langle a,x\rangle}) + c_6$ where $c_1, c_2, ..., c_6$ are the known constants.

For $q(x)$ of $\widetilde{C}^p(F)$ we denote by $B(q(x))$ the set of all invariants

$$I_j(a,\beta,q),\ \|q^a(x)\|,\ \text{for } j = 2,3,4;\ a \in S;\ \beta \in \Gamma_a, \tag{68}$$

where S is the set of all maximal elementsof Γ. The set $B(q(x))$ can be considered as the vector of the space l_2. To determine a polynomial of $E_0(N_k, M_k, S_k)$ we use some invariants of (68). Let O_1 and O_2 be the set of vectors a and pairs $\{a,\beta\}$ such that the invariants $\|q^a(x)\|$ and $I_j(a,\beta)$ $(j = 2,3,4)$ are used for finding the elements of $E_0(N_k, M_k, S_k)$. For every function $q(x)$ of $\widetilde{C}^p(F)$ we denote by $B_k(q(x))$ the set of invariants $\|q^a(x)\|$, for $a \in O_1$, and $I_j(a,\beta,q)$ $(j = 2,3,4)$ for $\{a,\beta\} \in O_2$. Denote by $E_0'(N_k, M_k, S_k)$ the set of all vectors $B_k(p(x))$, for $p(x) \in E_0(N_k, M_k, S_k)$. This set is a p_k'-dimensional subset of l_2 . Clearly $p_k' \sim p_k,\ p_k < S_k^3$.

¿From definition of invariants (68) we see that $B(q(x))$ continuously depends on q. Therefore the mapping B_k taking $q(x)$ to $B_k(q(x))$ is a continuous mapping from $\widetilde{C}^p(F)$ to $R^{p_k'}$. Its restriction $B_{k,0}$ on $\widetilde{E}_0(N_k, M_k, S_k)$ is a continuous mapping from $\widetilde{E}_0(N_k, M_k, S_k)$ onto $E_0'(N_k, M_k, S_k)$. In Sections 3, 4, we proved that the inverse mapping $B_{k,0}^{-1}$ exists. It is continuous on every compact subset of $E_0'(N_k, M_k, S_k)$. (Moreover every coordinate of $V(q)$ was obtained from coordinates of $B_k(q(x))$ by a finite number of arithmetical operations) Hence the mapping $B_{k,0}^{-1}$ from $E_0'(N_k, M_k, S_k)$ onto $\widetilde{E}_0(N_k, M_k, S_k)$ is continuous. Thus for every $q(x) \in \widetilde{E}_0(N_k, M_k, S_k)$ and $\epsilon_k > 0$ there is $\delta_k(q)$ such that

$$\|B_k(q) - B_k(\widetilde{q})\| < \delta_k(q)\ \Rightarrow \|q(x) - \widetilde{q}(x)\| < \epsilon_k \tag{69}$$

for all $\widetilde{q}(x) \in \widetilde{E}_0(N_k, M_k, S_k)$. By continuity of B_k on $\widetilde{C}^p(F)$ for every $q_k(x) \in \widetilde{E}_0(N_k, M_k, S_k)$ there is $\delta_k'(q_k)$ such that

$$\|q_k(x) - q(x)\| < \delta_k'(q_k)\ \text{ implies } \|B_k(q_k) - B_k(q)\| < \frac{1}{2}\delta_k(q_k) \tag{70}$$

for all $q(x) \in \widetilde{C}^p(F)$. Let $\widetilde{C}_0^p(F)$ be the set of all functions $q(x)$ of $\widetilde{C}^p(F)$ such that

$$q_k(x) \equiv \sum_{a \in G(N_k, M_k, S_k)} (q(x), e^{i\langle a,x\rangle}) e^{i\langle a,x\rangle} \tag{71}$$

is contained in $\widetilde{E}_0(N_k, M_k, S_k)$ and at least one of the inequalities $\|q(x) - q_k(x)\| < \delta_k'(q_k(x))$,

$$\|B_k(q_k) - B_k(q)\| < \frac{1}{2}\delta_k(q_k) \tag{72}$$

holds for all $k = 1, 2, ...,$. Suppose ϵ_k tends to zero as k tends to infinity.

Denote by $\widetilde{E}_1(N_k, M_k, S_k)$ the set of all $q(x) \in \widetilde{E}_0(N_k, M_k, S_k)$ such that (69) is satisfied for

$$\delta_k(q) = \frac{1}{N_k^{p-2}}, \ \epsilon_k = \frac{1}{\ln(\ln(N_k))}$$

and for all $\widetilde{q}(x) \in \widetilde{E}_0(N_k, M_k, S_k)$. Let $\widetilde{C}_1^p(F)$ be the set of all functions $q(x)$ of $\widetilde{C}^p(F)$ such that the corresponding partial sum (71) is contained in $\widetilde{E}_1(N_k, M_k, S_k)$ for all $k = 1, 2, ...,$. Direct calculations show that for all $q(x) \in C^p(F)$ we have

$$\|q(x) - q_k(x)\| < \frac{1}{N_k^p}, \ \|B_k(q) - B_k(q_k)\| < \frac{1}{2N_k^{p-2}}. \tag{73}$$

Theorem 13. *If $q(x) \in \widetilde{C}_0^p(F)$ or $q(x) \in \widetilde{C}_1^p(F)$ then $q(x)$ can be constructed by $B(q)$, hence by the Floquet spectrum of the operator $L(q)$. If $q(x)$, $\widetilde{q}(x) \in \widetilde{C}_0^p(F)$ or $q(x)$, $\widetilde{q}(x) \in \widetilde{C}_1^p(F)$ and $B(q) = B(\widetilde{q})$, (say, the operators $L(q)$, $L(\widetilde{q})$ have the same Floquet spectrum) then $q(x) = \widetilde{q}(x)$.*

Proof: Assume that $q(x) \in \widetilde{C}_0^p(F)$ and $B(q))$ is given. Using definition of $\widetilde{C}_0^p(F)$ and (70), (71), (72) we obtain that there is an element b_k of $E_0'(N_k, M_k, S_k)$ such that

$$\|B_k(q) - b_k\| < \frac{1}{2}\delta_k(q_k). \tag{74}$$

By b_k, we construct the polynomial $p_k(x) = B_{k,0}^{-1}(b_k)$. Then by (69), (70) and (74) we have

$$\|B_k(q_k) - B_k(p_k)\| < \delta_k(q_k), \ \|q_k(x) - p_k(x)\| < \epsilon_k. \tag{75}$$

Hence $q(x)$ is the limit of the sequence $\{p_k(x)\}$.

Now let $q(x)$ be in $\widetilde{C}_1^p(F)$. Then by (71) and (73) there is an element d_k of $E_1'(N_k, M_k, S_k)$ such that

$$\|B_k(q) - d_k\| < \frac{1}{2N_k^{p-2}}, \quad \|B_k(q_k) - d_k\| < \frac{1}{N_k^{p-2}} \tag{76}$$

By d_k we can construct a function $\widetilde{p}_k(x) = B_{k,0}^{-1}(d_k)$. Then the inequalities (69), (76) imply that (see definition of $\widetilde{E}_1(N_k, M_k, S_k)$)

$$\|q_k(x) - \widetilde{p}_k(x)\| < \frac{1}{\ln(\ln(N_k))}.$$

Hence $q(x)$ is the limit of the sequence $\{\widetilde{p}_k(x)\}$.

If $\widetilde{q}(x)$, $q(x)$ are in $\widetilde{C}_0^p(F)$ or in $\widetilde{C}_1^p(F)$ and $B(q) = B(\widetilde{q})$, then applying this reasoning to $\widetilde{q}(x)$ and $q(x)$, we see that both $\widetilde{q}(x)$ and $q(x)$ are the limit of $g_k(x)$ or $\widetilde{p}_k(x)$, where $g_k = B_{k,0}^{-1}(e_k)$ and $e_k \in E_0'(N_k, M_k, S_k)$ is chosen so that $\|B_k(q) - e_k\| < \frac{1}{2}\min(\delta_k(q_k), \delta_k(\widetilde{q}_k))$. Hence $\widetilde{q}(x) = q(x)$ ■

In this paper, we have shown how the potential can be constructed by spectral invariants. We do not analyse the sets $\widetilde{C}_0^p(F)$ and $\widetilde{C}_1^p(F)$. Analysis of these sets is very technical. It requires the additional investigations. We shall do it in following papers. Now we only note the following:

Remark 6: It can be proved that the set $\widetilde{E}_1(N_k, M_k, S_k)$ has asymptotically full measure on $\widetilde{E}(N_k, M_k, S_k)$. It means that the ratio of measures of the sets

$$V(\widetilde{E}_1(N_k, M_k, S_k)) \cap \{x \in R_+R^{p_k-4} : \|x\| < r\}, \ \{x \in R_+R^{p_k-4} : \|x\| < r\} \tag{77}$$

tends to 1 as k tends to infinity for fixed r and as r tends to infinity for fixed k (It means that the set $\widetilde{C}_1^p(F)$ has asymptotically full measure on $\widetilde{C}^p(F)$ in this sense). In other words, to obtain the first set from the second one in (77) we eliminate the set A which has asymptotically zero measure. The larger smoothness p is, the smaller the set A is. In the case of infinitely differentiable functions ($p = \infty$) the set A has measure zero. More precisely, if $p = \infty$ then A is a union of some surfaces and subspaces $z(a) = 0$. The separated potentials show that to determine the potential in $C^p(F)$ uniquely (modulo translations) from spectral invariants, it is necessary to eliminate some of these subspaces. Thus the sufficient condition to solve the inverse problem by these method is close to the necessary condition.

In Theorem 11 the set $\widetilde{C}_0^p(F)$ depend on sequences $\{(N_k, M_k, S_k)\}_{k=1}^{\infty}$. We redenote it by $\widetilde{C}_0^p(F, \{(N_k, M_k, S_k)\})$. Let

$$\widetilde{C}_2^p(F) \equiv \bigcup_{\{(N_k, M_k, S_k)\}} \widetilde{C}_0^p(F, \{(N_k, M_k, S_k)\}),$$

where the union is taken over all possible sequence $\{(N_k, M_k, S_k)\}$ of triples (N_k, M_k, S_k) satisfying (6). By Theorem 11 every $q(x) \in \widetilde{C}_2^p(F)$ can be constructively recovered by spectral invariants (68).To prove that $\widetilde{C}_2^p(F)$ has full measure on $\widetilde{C}^p(F)$ we have to analyse in detail the invariants used in section 3,4 and some other invariants. Nevertheless we construct some subset of $\widetilde{C}_2^p(F)$, more precisely, using Theorem 11 we find some sufficient conditions for recovability of the smooth potential. Suppose that

$$q(x) \in \widetilde{C}^p(F), q_k(x) \in E_0(N_k, M_k, S_k) \tag{78}$$

for some sequences $\{(N_k, M_k, S_k)\}$ satisfying the conditions:

a) N_1 is such a big number that $\mid z(a) \mid < \mid a \mid^{-p}$,for all a satisfying $\mid a \mid > N_1$. Note that there exsist such a number for every $q(x) \in C^p(F)$.

b) $N_k \gg P_{k-1}$, where $k = 2, 3, ...$,and P_{k-1} is a number satisfying $P_{k-1} > S_{k-1}, P_{k-1}^{5-p} < \frac{1}{2}\delta_{k-1}(q_{k-1})$. Here $\delta_{k-1}(q_{k-1})$ is defined as $\delta_k(q)$ in (69) , i.e.,

$$\|B_k(q_{k-1}) - B_k(\widetilde{q}_{k-1})\| < \delta_{k-1}(q_{k-1}) \ \Rightarrow \|q_{k-1}(x) - \widetilde{q}_{k-1}(x)\| < \epsilon_k$$

for all $\widetilde{q}_{k-1}(x) \in \widetilde{E}_0(N_{k-1}, M_{k-1}, S_{k-1})$.

Since $\Gamma = G(N_1, M_1, S_1) \cup (\cup_{k=2}^{\infty} (G(N_k, M_k, S_k) \backslash G(N_{k-1}, M_{k-1}, S_{k-1}))$ the Fourier series of $q(x)$ can be written in the form

$$q(x) = q_1(x) + \sum_{k=2}^{\infty} (q_k(x) - q_{k-1}(x)) = q_1(x) + \sum_{k=2}^{\infty} (q_{k,1}(x) + q_{k,2}(x))$$

where $q_{k,j}(x) = \sum_{a \in G_{k,j}} z(a) e^{i(a,x)}, j = 1,$
$G_{k,1} = \{a : \mid a \mid < P_{k-1}\} \backslash G(N_{k-1}, M_{k-1}, S_{k-1}),$
$G_{k,2} = G(N_k, M_k, S_k) \backslash \{a : \mid a \mid < P_{k-1}\}.$

Here the Fourier decomposition of $q_{k,2}(x)$ contains essentially more summand than the Fourier decomposition of $q_{k,1}(x)$ due to $\mid G_{k,2} \mid \gg \mid G_{k,1} \mid$ (see condition b)). By condition a) and by definition of P_{k-1} the Fourier coefficients $z(a)$ of $q_{k,2}(x)$ satisfy

$$\mid z(a) \mid < P_{k-1}^{-p}. \tag{79}$$

Corollary 14. *Suppose that $q(x)$ satisfies the condition (78) for a sequence $\{(N_k, M_k, S_k)\}$ satisfying a) and b). If for $a \in G_{k,1}$,$(k = 2, 3, ...,)$ the inequality (79) holds then the potential $q(x)$ can be constructively determined by spectral invariants $B(q)$. (Note that the last condition means that $q_{k,1}$ is as small as $q_{k,2}$)*

Proof. Using condition a) for $a \in G_{k,2}$ and (79) for $a \in G_{k,1}$ we obtain

$$\mid q^a(x) - q^a_{k-1}(x) \mid < P^{1-p+\varepsilon}_{k-1}, \ \forall x \in F, \tag{80}$$

where $\frac{1}{2} > \varepsilon > 0$, $q^a(x)$ and $q^a_{k-1}(x)$ are directional potentials for $q(x)$ and $q_{k-1}(x)$ respectively. If a is long maximal with respect $Q(N_{k-1}, M_{k-1}, S_{k-1})$, then

$$\mid q^a(x) - (z(a)e^{i\langle a,x\rangle} + z(-a)e^{i\langle -a,x\rangle}) \mid < P^{1-p+\varepsilon}_{k-1}, \ \forall x \in F$$

This together with (80) yield

$$\mid I_j(a, \beta, q) - I_j(a, \beta, q_{k-1}) \mid < P^{3-p+\varepsilon}_{k-1}, \mid\mid B_{k-1}(q(x)) - B_{k-1}(q_{k-1}(x)) \mid\mid < P^{5-p}_{k-1}$$

The last inequality, condition b) and (78) imply that

$$q(x) \in \widetilde{C}^p_0(F, \{(N_k, M_k, S_k)\}).$$

Thus, the corollary is a consequence of Theorem 11■

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